# Enumerating the Orthogonal Diagonal Latin Squares of Small Order for Different Types of Orthogonality

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**Abstract.** The article describes computational experiments aimed to enumerating the number of orthogonal diagonal Latin squares for general and special types of orthogonality. General type of orthogonality can be verified using Euler-Parker method, corresponding the number of main classes of orthogonal diagonal Latin squares, the number of normalized orthogonal diagonal Latin squares and total number of orthogonal diagonal Latin squares of general type form previously unknown numerical series A330391, A305570 and A305571 (calculated up to order 8) has been added to OEIS. Self-orthogonal (SODLS), doubly self-orthogonal (DSODLS) and extended self-orthogonal diagonal Latin squares (ESODLS) form the set of special types of orthogonality. For each of these types corresponding numerical series was calculated and published in OEIS with numbers A329685, A287761, A287762 (SODLS, up to order 10), A333366, A333367, A333671 (DSODLS, up to order 10) and A309210, A309598, A309599 (ESODLS, up to order 8). Values for orders 1–8 were obtained by analyzing the complete lists of canonical forms of the main classes of orthogonal DLSs obtained by the authors by exhaustive search. Values for order 9 were derived from the SODLS list of order 9 provided by Harry White. Values for order 10 were obtained by analyzing the list of SOLS of order 10, available online (van Vuuren et al.). The values obtained confirm the similar values for SODLS and DSODLS obtained previously by Francis Gaspalou and partially published by Harry White. For some of the obtained numerical values previously unknown mathematical relations are empirically established.

**Keywords:** Desktop grid · Volunteer computing · BOINC · Combinatorics · Orthogonal diagonal Latin squares · Self-orthogonal diagonal Latin squares · Doubly self-orthogonal diagonal Latin squares · Integer sequences · OEIS · Gerasim@Home

## 1 Introduction

Latin squares (*abbr.* LS) are a well-known type of combinatorial objects the study of which is devoted to a fairly large number of scientific papers [1,2]. Diagonal


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Latin squares (*abbr.* DLS) are a special case of LS where in addition to the difference in the values of the elements in the rows and columns, an additional restriction is imposed on the difference in the values of the elements on the diagonals (or, in other words, the sets of elements forming the diagonals must be transversals). Pair of squares $A$ and $B$ is called orthogonal (*abbr.* OLS or ODLS) if all ordered pairs of values $(a_{ij}, b_{ij})$, $i, j = \overline{1, N}$, where $N$ – order of square, are different. Latin squares are used in such fields of science as experiment planning, error correction codes, cryptography, construction of schedules of a certain kind. They have a close relationship with magic squares.

The classic approach to checking LS/DLS for the presence of OLS/ODLS is the Euler-Parker approach [3] based on getting transversals (diagonal transversals) set [4,5] and then finding a subset of $N$ disjoint transversals. In practice this approach most efficiently works using software implementation of the dancing link algorithm (*abbr.* DLX) [6,7]. The search pace in this case is limited precisely by the DLX algorithm.

There are a set of open questions associated with the OLS/ODLS: problems of enumerative combinatorics, classification of the combinatorial structures (graphs) from the ODLS on the set of orthogonality binary relation [8–10]. The most famous unsolved problem in this area is the problem of finding a triple (or pseudotriples) of mutually orthogonal LS/DLS (*abbr.* MOLS/MODLS) of order 10, which despite the numerous attempts at construction was not found, but it has not been conclusively proven that it does not exist.

## 2 The Concept of Orthogonality of General and Special Types

In enumerative combinatorics there are a number of problems associated with counting the number of combinatorial objects of a certain type depending on its dimension (for example, the number of permutations with specified properties [11], the number of graph isomorphism classes [12], etc.). Typically, as a result of solving the problem a numerical series is obtained, the $N$-th value of which determines the number of corresponding objects of dimension $N$. These numerical series are of fundamental interest and have a number of practical applications. A large number of such series (more than 300,000) is a part of the Online Encyclopedia of Integer Sequences (*abbr.* OEIS) [13].

The team of authors developed a highly efficient generator of DLS of a given order $N$ based on a Brute-Force method and the principle of varying the order of filling the square cells and software implementation based on nested loops (without recursion) and bit arithmetic. Using it within the volunteer distributed computing project Gerasim@Home[1] on BOINC platform [14,15] a direct enumeration of all DLS orders up to $N = 9$ were organized, the search pace for $N = 10$ using this generator amounted to about 6–7 million DLS/s for a single-threaded CPU implementation. It took about 3 months to enumerate all DLSs of

[1] http://gerasim.boinc.ru.

order $N = 9$ (using grid within Gerasim@Home project and, regardless of this, on academician Matrosov computing cluster) [16]. As a result of the calculations, the numerical series "the number of DLS of order $N$ with a fixed first row" and "the number of DLS of order $N$" were obtained and included in the OEIS under numbers A274171 and A274806.

Similar numerical series for ODLS which are rarer objects than DLS were unknown at the time of the start of scientific research (the number of OLS is known up to order 9 [17]).

As already noted above, the determination of the presence or absence of ODLS for a given DLS is efficiently possible using the Euler-Parker method by reducing the source problem to the exact cover problem and solving it using the DLX algorithm and its software implementation [6,7]. In this case the search pace is limited by the DLX software implementation and amounts to about 1,000–2,000 DLS/s for the order $N = 10$, i.e. 3 orders of magnitude lower than the above pace of the DLS generators (it would be nice to make a direct ODLS generator, however, at present the effective implementation of this idea is not known). When searching for ODLS within the Gerasim@Home project, the effective search pace was additionally increased to 7,000–8,000 DLS/s by using the canonization procedure which reduced to finding symmetrically placed transversals in the LS and setting them to the place of the main and back diagonals by rearranging rows and columns of square [18], combined with a check for the presence of ODLS.

The processing pace can be significantly increased if we get away from the use of transversals and the DLX algorithm. In the general case this is impossible, but in some special cases this idea has a right to life. For example, in the RakeSearch[2] volunteer distributed computing project all possible ODLS pairs of order 9 were found where the square $B$, orthogonal to the checked square $A$, was obtained by rearranging the rows of square $A$ (special case of ESOLS). At the same time the search pace is approximately an order of magnitude higher (in the region of 70,000–80,000 DLS/s), which made it possible to discover several dozen new combinatorial structures from DLSs of order 9[3], the ODLS to which were found using the Euler-Parker method during post-processing of row-permutation pairs of ODLS [9]. Another well-known type of ODLS, also formed without the use of transversals, is a self-orthogonal LS/DLS (*abbr.* SOLS/SODLS) [19], where the orthogonal square $B$ of a pair of ODLS is obtained by transposing square $A$. For example, the square below is SODLS of order 10:

[2] https://rake.boincfast.ru/rakesearch/.
[3] http://evatutin.narod.ru/evatutin_ls_all_structs_n9_eng.pdf.

$$\begin{pmatrix} 0&1&2&3&4&5&6&7&8&9\\ 5&2&0&9&7&8&1&4&6&3\\ 9&5&7&1&8&6&4&3&0&2\\ 7&8&6&4&9&2&5&1&3&0\\ 8&9&5&0&3&4&2&6&7&1\\ 3&6&9&5&2&1&7&0&4&8\\ 4&3&1&7&6&0&8&2&9&5\\ 6&7&8&2&5&3&0&9&1&4\\ 2&0&4&6&1&9&3&8&5&7\\ 1&4&3&8&0&7&9&5&2&6 \end{pmatrix}.$$

SODLS search can be organized at an efficient pace that is several orders of magnitude faster than the Euler-Parker method, both due to the fact that transposing a square and subsequent checking a pair of squares for orthogonality are very fast operations, and due to the possibility of developing a specialized highly efficient ODLS generator of corresponding type based on X-based diagonal fillings, principles of variation of filling cells order and specialized software implementation with nested loops and bit arithmetic [16]. In addition to self-orthogonal squares, some articles also mention doubly self-orthogonal squares (*abbr.* DSOLS/DSODLS) [20], where in addition to the requirement of orthogonality from transposition, an additional requirement is imposed on the presence of an orthogonal square after the transposition of the square from the back diagonal. DSOLS/DSODLS are a special kind (subset) of SOLS/SODLS. An example of such DSODLS of order 9 is given below:

$$\begin{pmatrix} 0&1&2&3&4&5&6&7&8\\ 2&4&3&0&7&6&8&1&5\\ 4&6&7&1&8&2&3&5&0\\ 8&3&5&6&0&7&1&2&4\\ 7&8&1&4&5&3&0&6&2\\ 3&7&0&2&1&8&5&4&6\\ 1&5&4&7&6&0&2&8&3\\ 5&0&6&8&2&1&4&3&7\\ 6&2&8&5&3&4&7&0&1 \end{pmatrix}.$$

When working with Latin squares, it is necessary to take into account an isomorphism which allows in some cases to reduce the search space significantly (by several orders of magnitude) and the corresponding computational costs for practical software implementation of algorithms. We will say that DLSs belong to the same main class if they have the same canonical forms (*abbr.* CF) [21] – lexicographically minimal string representations for the corresponding squares within the indicated classes of isomorphism. All DLSs within the main class can be obtained by applying a combination of a number of equivalent transformations [22] and are characterized by identical properties (the number of transversals, the presence of ODLS, the number and composition of generalized symmetries (automorphisms), etc.).

The above definition of self-orthogonality can be extended to the class of ODLS where corresponding DLSs within the pair of orthogonal squares have the same main class. We will call such type of orthogonal squares as ESODLS (Extended SODLS). Their search can also be effectively organized without the use of transversals, and their properties are of theoretical interest (for example, all cliques from ODLS of orders 1–8 with a cardinality of more than 2 and most of the currently known cliques of order 9 includes ESODLS [23]). SODLS is a special case of ESODLS by definition. An example of ESODLS of order 10 is shown below:

$$\begin{pmatrix} 0&1&2&3&4&5&6&7&8&9 \\ 1&2&0&6&7&9&8&3&4&5 \\ 3&6&7&9&8&4&2&5&1&0 \\ 4&0&8&5&2&3&7&1&9&6 \\ 5&9&4&8&3&6&0&2&7&1 \\ 7&8&6&4&0&1&3&9&5&2 \\ 6&4&5&2&1&7&9&0&3&8 \\ 9&5&1&7&6&0&4&8&2&3 \\ 2&3&9&0&5&8&1&4&6&7 \\ 8&7&3&1&9&2&5&6&0&4 \end{pmatrix}.$$

At the start of the study the number of orthogonal DLSs was calculated only for small-order SODLS and DSODLS by Francis Gaspalou[4], and its results (including DLS lists) were either not published at all or were known only in private correspondence, which makes actual the task of independent verification of the relevant values for SODLS and DSODLS, as well as a similar calculation of the unknown number of small order ESODLS and ODLS and publication corresponding lists of CFs and numerical series in OEIS.

## 3 Enumerating the ODLS of General Type

The search for all ODLS of orders 1–7 using the software developed by the authors was performed on the single PC, for order 8 the organization of the corresponding computational experiment in the Gerasim@Home project was required. As a result of the calculations complete lists of ODLS CFs of orders 1–8 were obtained. Based on these lists, we can obtain both special type ODLS lists (DSODLS/SODLS/ESODLS) by dropping some of the corresponding ODLS CFs with not interesting properties, and expand the corresponding lists on the DLSs with the ordered first row (also called as normalized DLS) and on the general type DLSs. For orders more than 8, obtaining complete lists of CFs of the ODLS is difficult due to their large number and the huge computational costs required for this. The obtained numbers of the ODLS main classes, the numbers of ODLS with a fixed first row and the numbers of general type ODLS of orders 1–8 are given in Tables 1, 2 and 3 (5th column) and were added to the OEIS

[4] http://www.gaspalou.fr/magic-squares/index.htm.

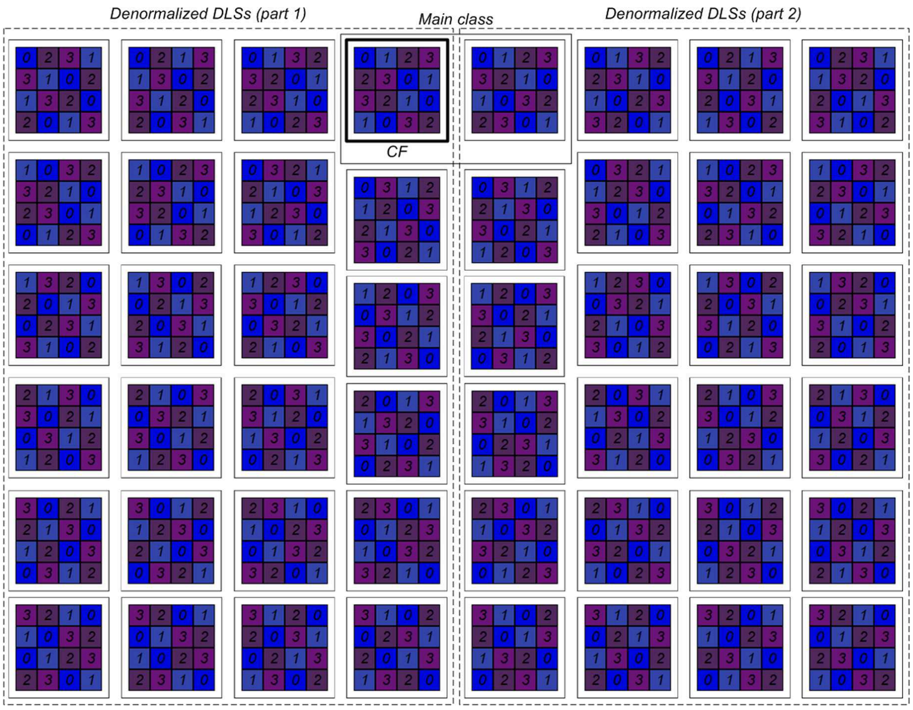


**Fig. 1.** An example of obtaining isomorphic squares from CF (top middle in a bold frame). A thin line picked out the corresponding main class from 2 normalized by the first row DLSs obtained by applying equivalent transformations to CF. Each of the normalized DLS from the main class forms a subclass of 1 normalized and the remaining $4! - 1 = 24 - 1 = 23$ denormalized DLSs (dashed rectangles on the left and on the right sides). In total, the canonical form corresponds to the isomorphism class from 48 general type DLSs.

under the numbers A330391, A305570 and A305571. Schematically the process of obtaining squares of various types from the CF is shown in Fig. 1.

## 4 Enumerating the SODLS, DSODLS and ESODLS

Based on the data of the ODLS CFs lists we can determine the number of main classes of SODLS of orders $1 \leq N \leq 8$. Determining the number of SODLS main classes of orders $N > 8$ in this way is not possible due to the lack of similar lists of ODLS CFs. However, the SODLS list of order 9 is known which was provided by Harry White[5] at the request of the authors and includes 224,832 DLSs with an ordered first row. Based on it, it is possible to find corresponding different CFs and calculate their number which turned out to be equal to 470 which is the 9th member of the corresponding numerical series.

[5] http://budshaw.ca/SODLS.html.

The obtained list of SODLS CFs of order 9, in its turn, made it possible to explore new combinatorial structures in addition to the previously obtained structures after analysis of central symmetry properties [24] and row-permutation ODLS from RakeSearch project [9]:

- $4N6M1C$,
- $6N5M3C$,
- $6N5M3C2$,
- $6N8M3C$,
- $8N8M4C$,
- $10N9M4C$,
- $10N9M4C2$,
- $10N19M3C$,
- $10N21M3C$,
- $12N20M6C2$,
- $12N20M6C3$,
- $12N21M2C$,
- $14N22M7C$,
- $18N61M4C$,
- $20N28M9C$,
- $32N42M12C$,
- $36N80M14C$,
- $96N402M8C$,
- $98N470M21C$,
- $98N502M21C$,
- $162N606M37C$,
- $540N1500M11C$,
- $760N944M57C$.

SOLS list of order 10 is also known [25]. By canonizing it, we can get a list of SODLS of order 10 including 30,502 different CFs, which is the 10th member of the investigated numerical series (these lists, identical in composition, were obtained by the co-authors of the article independently from each other using various software implementations, which is an independent confirmation of the correctness of the obtained values). The obtained numerical values form a numerical series (see Table 1, 3rd column) and were added to OEIS under the number A329685.

Having lists of SODLS CFs, by applying equivalent transformations to each CF in their composition, we can get the full main classes from the DLS corresponding to them, sum their cardinalities and get the total number of normalized SODLS, which is the next numerical series (see Table 2, 3rd column) has been added to OEIS with number A287761.

An interesting feature of the obtained series is the fact that all SODLS main classes of order 10 are full-sized: they are formed by 15,360 normalized DLSs each and include 7,680 SODLS and 7,680 ESODLS with orthogonality by transposition from the back diagonal. This feature was identified empirically in the course of a computational experiment where all SODLS main classes were constructed and their cardinalities evaluated. For dimensions $1 \leq N \leq 9$, this property is not fulfilled, and different main classes contain a different number of normalized DLSs, apparently, due to the presence of generalized symmetries (automorphisms) in the corresponding SODLS [26], which reduces the cardinalities of some main classes in comparison with the theoretical maximum (see A299784 and A299787 series in OEIS).

Multiplying the obtained numerical values by $N!$, we can get the total number of SODLS (see Table 3, 3rd column), forming the sequence A287762 that also has been added to OEIS.

Having ready lists of SODLS main classes, normalized SODLS and SODLS of general type we can get the corresponding numerical series for DSODLS by performing an additional check for the presence of an orthogonal square when transposing the corresponding SODLS from the back diagonal. In this case we can get the following numerical series, also not presented in the OEIS at the time of the research (see Tables 1, 2 and 3, 2nd columns in each table). Currently, they have also been tested and added to OEIS under the numbers A333366, A333367 and A333671.

It is easy to notice that DSODLS of order 10 do not exist (previously a similar result was obtained for DSOLS of order 10 [20]). The latter property, in particular, implies a meager set of combinatorial structures formed by SODLS of order 10 compared to, for example, combinatorial structures obtained in the neighborhoods of generalized symmetries [26] or from SODLS of lower orders (for example, of order 9).

During the publication of preliminary information related to the calculations mentioned above, it turned out that similar values for SODLS and DSODLS were obtained earlier by Francis Gaspalou and partially published by Harry White. Thus, the results of calculations performed within the Gerasim@Home project with subsequent post-processing can be an independent confirmation of the correctness of the previously found values.

The calculation of the ESODLS number can be performed similarly to the considered above: firstly, for each of the ODLS CFs of a given order it is necessary to find the corresponding main classes. Then, the resulting main classes may be expanded by a combination of equivalent transformations to normalized ESODLS and then by changing the values of the elements to general type ESODLS.

For order $N = 10$ a set of 33,240 ESODLS CFs obtained in the Gerasim@Home project through a targeted search of ESODLS is currently known (33,238 CFs of them were obtained by searching for cell mapping schemes (*abbr.* CMS) with a canonical loops structure $\{1{:}10, 2{:}45\}$ (where, among other things, all SODLS are included) and 2 more CFs were obtained during the study of the neighborhoods of generalized symmetries, they correspond to CMSs with the loops structure $\{4{:}25\}$ and combinatorial structures of the "loop-4" type [10] including 4 DLSs with the same CF). During the computational experiments that are currently ongoing, other ESODLS CFs are not obtained: they either do not

**Table 1.** ODLS main classes of different type of orthogonality

| ODLS type DLS order $N$ | DSODLS (A333366) | SODLS (A329685) | ESODLS (A309210) | ODLS (A330391) |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 |
| 7 | 2 | 2 | 5 | 5 |
| 8 | 8 | 8 | 23 | 1,105 |
| 9 | 88 | 470 | ? | ? |
| 10 | 0 | 30,502 | $\geq$ 33,240 | ? |

**Table 2.** Normalized ODLS of different type of orthogonality

| ODLS type DLS order $N$ | DSODLS (A333367) | SODLS (A287761) | ESODLS (A309598) | ODLS (A305570) |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 2 | 2 | 2 | 2 |
| 5 | 4 | 4 | 4 | 4 |
| 6 | 0 | 0 | 0 | 0 |
| 7 | 64 | 64 | 256 | 256 |
| 8 | 1,152 | 1,152 | 4,608 | 632,064 |
| 9 | 28,608 | 224,832 | ? | ? |
| 10 | 0 | 234,255,360 | $\geq$ 510,566,400 | ? |

exist or are very rare. The resulting value allows us to impose a lower limit on the total number of ESODLS main classes of order 10 which is very close to the exact value. All currently known ESODLS main classes (as well as SODLS) are full-sized, which was established empirically during the calculation of the cardinalities of each of them within the performed computational experiment.

Corresponding numerical series (see Tables 1, 2 and 3, 4th columns) were also added to the OEIS under the numbers A309210, A309598 and A309599.

**Table 3.** General case ODLS of different type of orthogonality

| ODLS type<br>DLS order $N$ | DSODLS<br>(A333671) | SODLS<br>(A287762) | ESODLS<br>(A309599) | ODLS<br>(A305571) |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 48 | 48 | 48 | 48 |
| 5 | 480 | 480 | 480 | 480 |
| 6 | 0 | 0 | 0 | 0 |
| 7 | 322,560 | 322,560 | 1,290,240 | 1,290,240 |
| 8 | 46,448,640 | 46,448,640 | 185,794,560 | 25,484,820,480 |
| 9 | 10,381,271,040 | 81,587,036,160 | ? | ? |
| 10 | 0 | 850,065,850,368,000 | $\geq$ 1,852,743,352,320,000 | ? |

All found numerical series turned out to be new, no coincidence with known numerical series for other types of combinatorial objects was revealed. All the found lists of ODLS CFs of general and special types are available online[6].

Taking into consideration the fact that DSODLS $\subseteq$ SODLS $\subseteq$ ESODLS $\subseteq$ ODLS, we can formulate the following relations between the numerical values found:

$$A333366(n) \leq A329685(n) \leq A309210(n) \leq A330391(n),$$
$$A333367(n) \leq A287761(n) \leq A309598(n) \leq A305570(n),$$
$$A333671(n) \leq A287762(n) \leq A309599(n) \leq A305571(n).$$

In addition, there is a number of interesting relations between the numerical values found. All of them are found empirically and currently do not have a theoretical explanation.

1. All ODLSs of orders 1, 4, and 5 are simultaneously DSODLS, SODLS and ESODLS; there are no other types of ODLSs for these orders.
2. All ODLSs up to order 7 are ESODLS.
3. All SODLS up to order 8 are DSODLS.
4. The number of normalized SODLS and general type SODLS of orders 7 and 8 is exactly 4 times less than the corresponding numbers of ESODLS of the same type and the same dimension:

$$A287761(n) \cdot 4 = A309598(n), n = \overline{7, 8},$$
$$A287762(n) \cdot 4 = A309599(n), n = \overline{7, 8}.$$

[6] http://evatutin.narod.ru/evatutin_odls_1_to_8.zip (ODLS CFs),
http://evatutin.narod.ru/evatutin_esodls_1_to_8.zip (ESODLS CFs),
http://evatutin.narod.ru/evatutin_sodls_1_to_10.zip (SODLS CFs),
http://evatutin.narod.ru/evatutin_dsodls_1_to_10.zip (DSODLS CFs).

For the SODLS and ESODLS main classes of orders 7 and 8 similar relations do not revealed. The similar ODLS/SODLS and SODLS/DSODLS values are also fractional.

## 5 Conclusion

Thus, using volunteer distributed computing, new numerical series were found (for ODLS and ESODLS) and the validity of previously partially known numerical series (for SODLS and DSODLS) was confirmed. All of them added to OEIS, a well-known resource in this field of knowledge. The obtained results prove the efficiency of distributed computing in enumerative combinatoric problems.

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