Southwest State University

Current state of research of the properties of Latin squares within the RakeSearch volunteer distributed computing project

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Kursk, 2026



Introduction



LS and DLS definition

$A = \|a_{ij}\|$

$i, j = \overline{1, N}$

$N = |S|$

$S = \{0, 1, 2, ..., N-1\}$

$$\forall i, j, k = \overline{1, N}, j \neq k : (a_{ij} \neq a_{ik}) \wedge (a_{ji} \neq a_{ki})$$

$$\forall i, j = \overline{1, N}, i \neq j : (a_{ii} \neq a_{jj}) \wedge (a_{N-i+1, N-i+1} \neq a_{N-j+1, N-j+1})$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\
2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\
3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\
4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\
5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\
6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\
7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\
8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\
9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1
\end{pmatrix}$$

Normalized LS of order 10

$$N! \times (N-1)!$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\
8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\
2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\
5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\
9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\
4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\
3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\
6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\
1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4
\end{pmatrix}$$

Normalized DLS of order 10

$$(N-1)!$$



Numerical characteristics of LS/DLS

Fast computable:

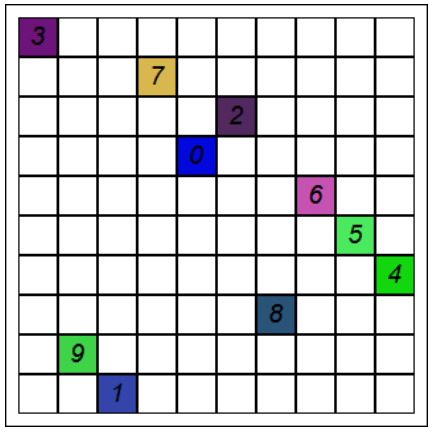
- intercalates number;
- transversals (general type or diagonal) number;

Relatively fast to compute:

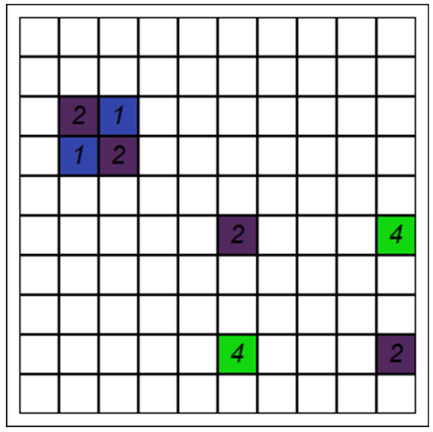
- OLS/ODLS number.

Computable intensive:

- full/short loops number;
- trivial/nontrivial Latin subrectangles number;
- main class cardinality;
- …



Transversal example



Intercalates example

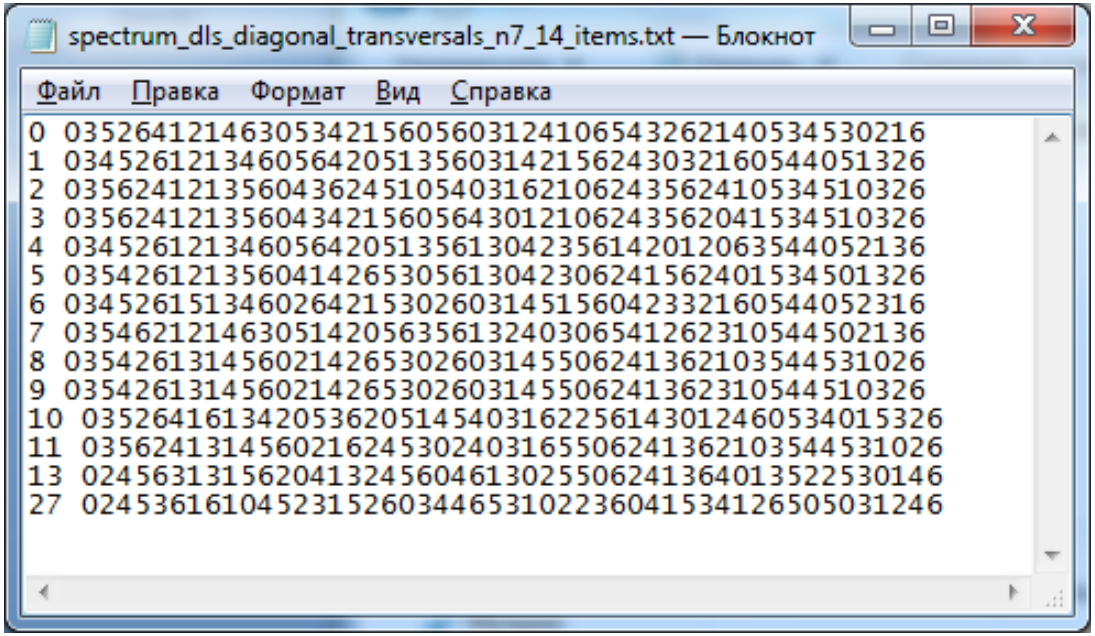
Numerical characteristic spectrum definition

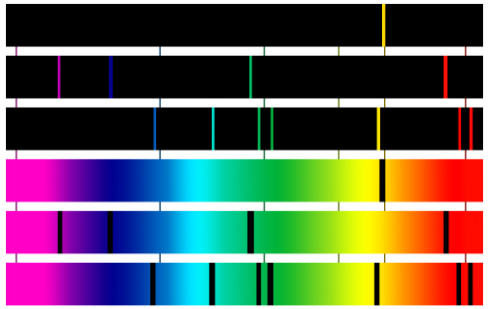
Spectrum S — a set of numerical values of a selected numerical characteristic X of combinatorial objects of a given type.

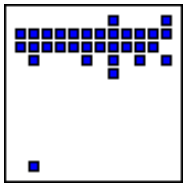
Example:

- type of combinatorial objects — DLS of order 7;
- numerical characteristic — diagonal transversals number.

$$S_{dt,\,DLS7} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 27\}$$







http://evatutin.narod.ru/spectra/spectrum_dls_diagonal_transversals_n7_14_items.txt

Numerical series in OEIS

A345760 a(n) is the number of distinct numbers of intercalates of order n diagonal Latin squares.

0, 0, 0, 1, 2, 1, 21, 61, 64

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,5

COMMENTS a(n) <= A307164(n) - A307163(n) + 1.
a(n) <= A287764(n).
a(10) >= 98, a(11) >= 145, a(12) >= 259, a(13) >= 200, a(14) >= 362, a(15) >= 536, a(16) >= 792, a(17) >= 685, a(18) >= 535, a(19) >= 447, a(20) >= 1011, a(21) >= 747, a(22) >= 872, a(23) >= 885, a(24) >= 1610, a(25) >= 1677, a(26) >= 1266, a(27) >= 1337, a(28) >= 2795. - Eduard I. Vatutin, Oct 02 2021, updated Mar 02 2025

LINKS Table of n, a(n) for n=1..9.
Eduard I. Vatutin, About the spectra of numerical characteristics of diagonal Latin squares of orders 1-7 (in Russian).
Eduard I. Vatutin, About the spectra of numerical characteristics of diagonal Latin squares of order 8 (in Russian).
Eduard I. Vatutin, About the approximation of spectra of numerical characteristics of diagonal Latin squares of order 9 (in Russian).
Eduard I. Vatutin, About the results of experiment with spectra of diagonal Latin squares using Brute Force and distributed computing projects Gerasim@Home and RakeSearch (in Russian).
Eduard I. Vatutin, Graphical representation of the spectra.
Eduard I. Vatutin, Proving lists (1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26).
E. I. Vatutin, N. N. Nikitina, M. O. Manzuk, I. I. Kurochkin, A. M. Albertyan, A. V. Kripachev, A. I. Pykhtin, Methods for getting spectra of fast computable numerical characteristics of diagonal Latin squares, Cloud and distributed computing systems in electronic control conference, within the National supercomputing forum (NSCF - 2022). Pereslavl-Zalessky, 2023. pp. 19-23. (in Russian)
E. I. Vatutin, V. S. Titov, A. I. Pykhtin, A. V. Kripachev, N. N. Nikitina, M. O. Manzuk, A. M. Albertyan and I. I. Kurochkin, Estimation of the Cardinalities of the Spectra of Fast-computable Numerical Characteristics for Diagonal Latin Squares of Orders N>9 (in Russian) // Science and education in the development of industrial, social and economic spheres of Russian regions. Murom, 2022. pp. 314-315.
E. I. Vatutin, V. S. Titov, A. I. Pykhtin, A. V. Kripachev, N. N. Nikitina, M. O. Manzuk, A. M. Albertyan, I. I. Kurochkin, Heuristic method for getting approximations of spectra of numerical characteristics for diagonal Latin squares, Intellectual information systems: trends, problems, prospects, Kursk, 2022. pp. 35-41. (in Russian)
Index entries for sequences related to Latin squares and rectangles.

EXAMPLE For n=7 the number of intercalates that a diagonal Latin square of order 7 may have is 0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 22, 26, or 30. Since there are 21 distinct values, a(7)=21.

CROSSREFS Cf. A287764, A307163, A307164, A309344, A344105, A345370, A345761.
Sequence in context: A051492 A380220 A164827 * A213976 A330354 A200859
Adjacent sequences: A345757 A345758 A345759 * A345761 A345762 A345763

KEYWORD nonn,more,hard

AUTHOR Eduard I. Vatutin, Jun 26 2021

EXTENSIONS a(9) added by Eduard I. Vatutin, Oct 22 2022

STATUS approved

- https://oeis.org
- more information: https://youtu.be/rAWfBZqwCl4

General strategy for heuristic spectra construction

- getting base spectrum;
- expanding of the base spectrum.

Base spectra:

- from different spectra for DLS (for example, construction of the transversals spectrum after diagonal transversals spectrum);
- from DLS of some special type (Brown, Gergely, (semi)cyclic, …).

Expanding of the base spectrum methods:

- neighborhoods after 1 intercalate/loop rotation;
- diagonalizing (for diagonal transversals spectra).



LS of order N to LS of order N+1 transformation



From LS N to LS N+1 transformation

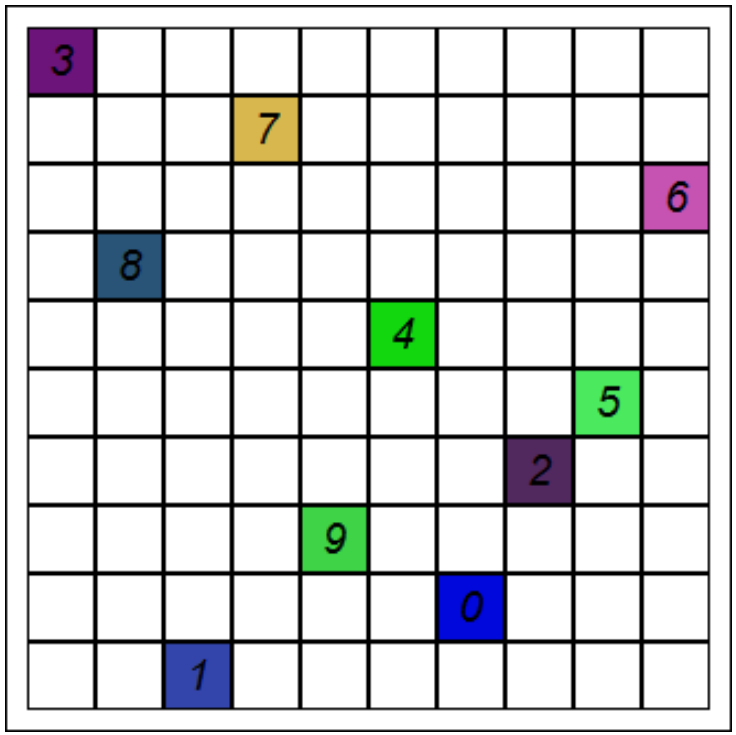
3	2	8	4	6	7	1	0	9	5
8	1	2	7	4	6	3	5	0	9
1	5	0	9	8	2	4	3	7	6
6	8	5	2	0	9	7	1	4	3
9	0	7	1	5	4	2	6	3	8
4	3	9	0	1	8	6	7	5	2
0	6	3	8	7	5	9	2	1	4
5	7	6	3	9	1	8	4	2	0
7	9	4	5	2	3	0	8	6	1
2	4	1	6	3	0	5	9	8	7



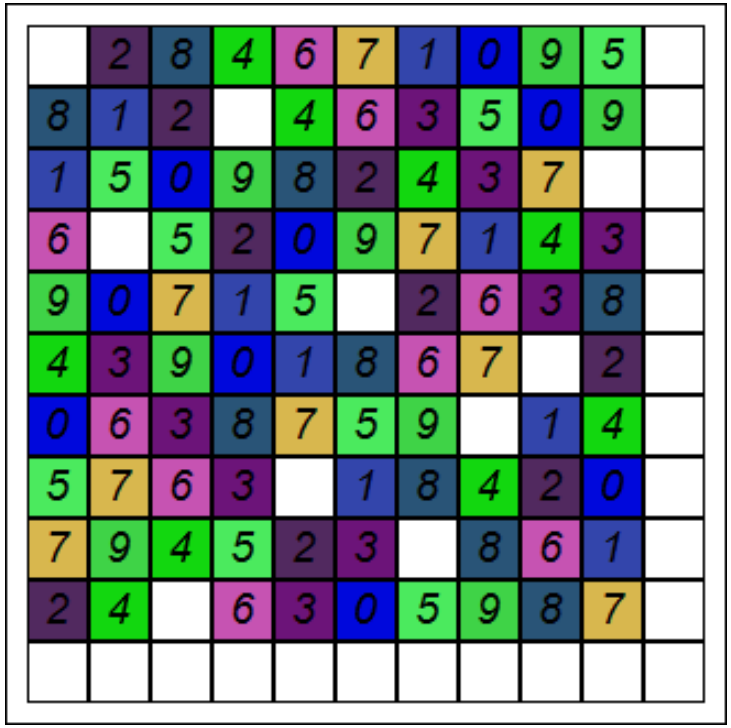
From LS N to LS N+1 transformation

3	2	8	4	6	7	1	0	9	5	A
8	1	2	7	4	6	3	5	0	9	A
1	5	0	9	8	2	4	3	7	6	A
6	8	5	2	0	9	7	1	4	3	A
9	0	7	1	5	4	2	6	3	8	A
4	3	9	0	1	8	6	7	5	2	A
0	6	3	8	7	5	9	2	1	4	A
5	7	6	3	9	1	8	4	2	0	A
7	9	4	5	2	3	0	8	6	1	A
2	4	1	6	3	0	5	9	8	7	A
A	A	A	A	A	A	A	A	A	A	

From LS N to LS N+1 transformation



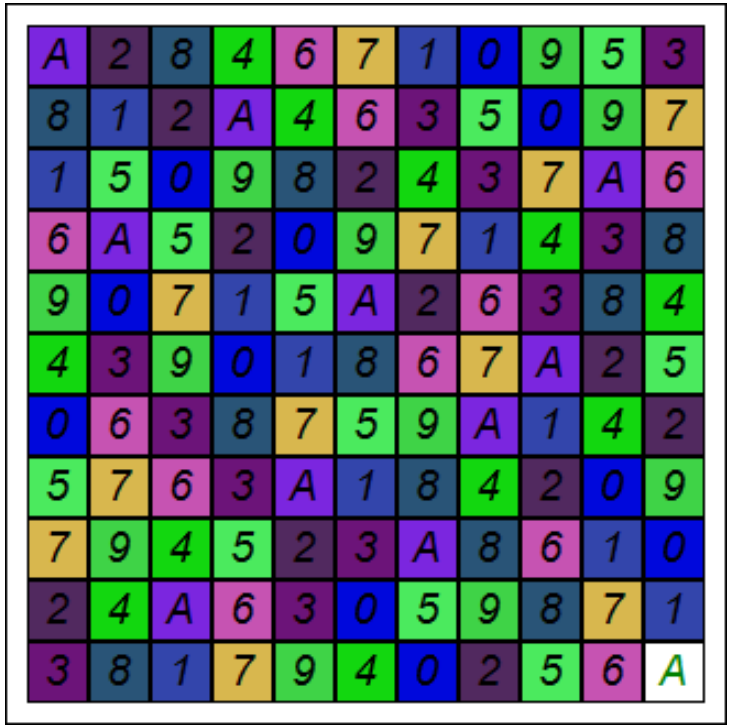
From LS N to LS N+1 transformation



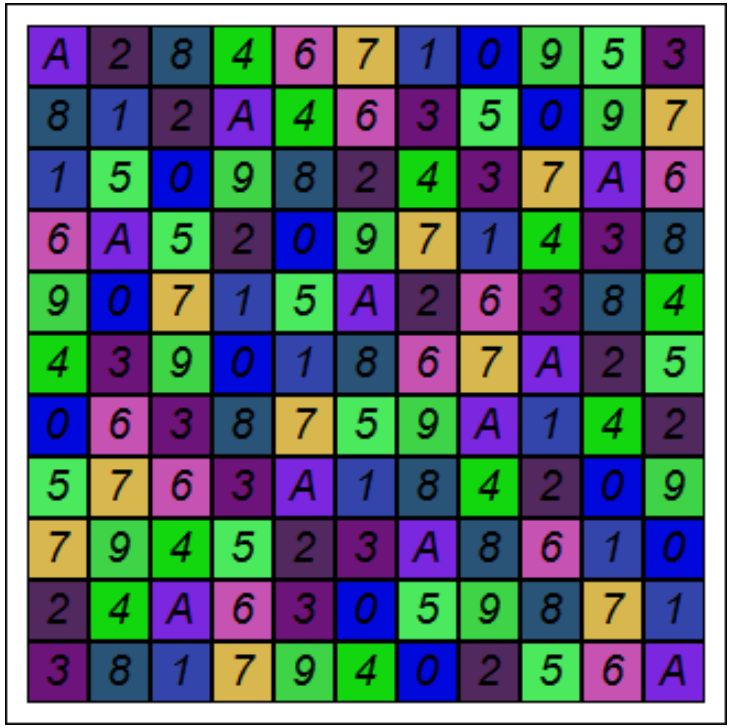
From LS N to LS N+1 transformation

A	2	8	4	6	7	1	0	9	5	3
8	1	2	A	4	6	3	5	0	9	7
1	5	0	9	8	2	4	3	7	A	6
6	A	5	2	0	9	7	1	4	3	8
9	0	7	1	5	A	2	6	3	8	4
4	3	9	0	1	8	6	7	A	2	5
0	6	3	8	7	5	9	A	1	4	2
5	7	6	3	A	1	8	4	2	0	9
7	9	4	5	2	3	A	8	6	1	0
2	4	A	6	3	0	5	9	8	7	1
3	8	1	7	9	4	0	2	5	6	

From LS N to LS N+1 transformation



From LS N to LS N+1 transformation



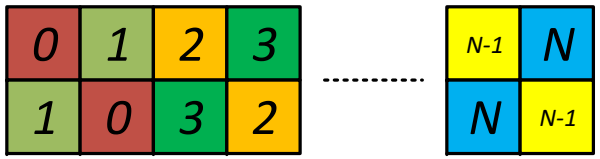
Heinrich-Wallis limit

Upper bound for intercalates number in LS

$$I_{\max}(N) \le \frac{N^2(N-1)}{4}, \quad N = 2k$$

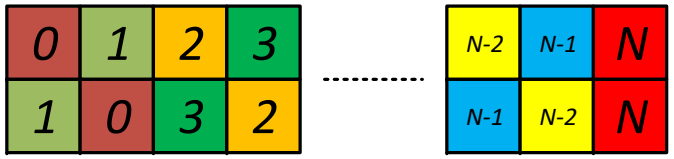
$$I_{\max}(N) \le \frac{N(N-1)(N-3)}{4}, \quad N = 2k+1$$

Even order N=2k

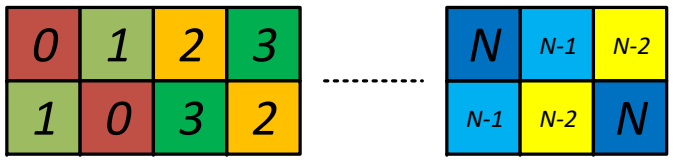


I(N) = N/2

Odd order N=2k+1



Wrong!



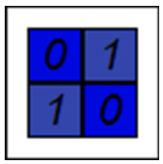
Right *I(N) = (N-3)/2*

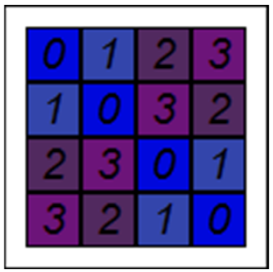
If all rows (unordered pairs) produced same: *x N(N-1)/2*

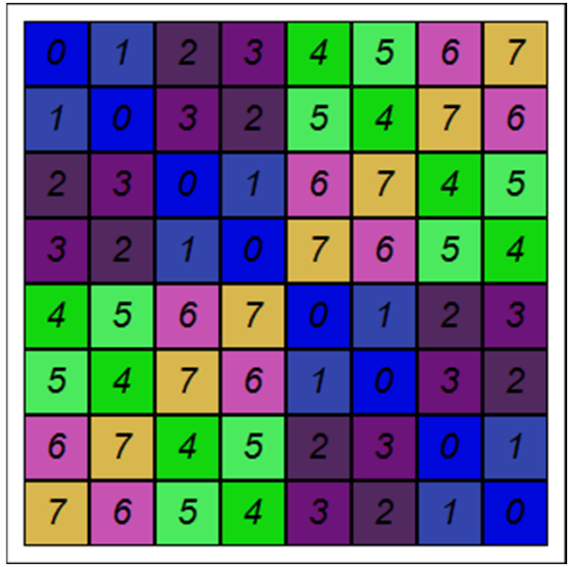
- K. Heinrich, W. Wallis, 1980

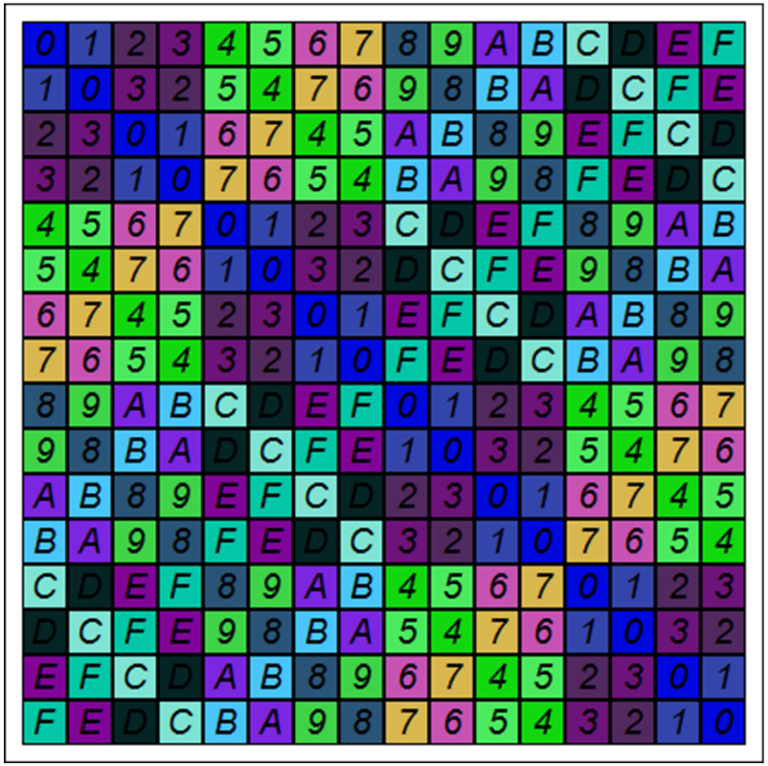
Some new theoretical results

- maximum number a(N) of intercalates in LS of order N = 2^n
  - a(N) = N^2*(N-1)/4 (K. Heinrich, W. Wallis, 1980)
  - How? For what type of squares?
  - Now I know: composite squares method 2x2 → 4x4 → 8x8 → … (**formula confirmed empirically!**)
  - Also: a(n) = n^2*(n-1)/4 **= A016152(log2(n)), new interconnection between different numerical series!**



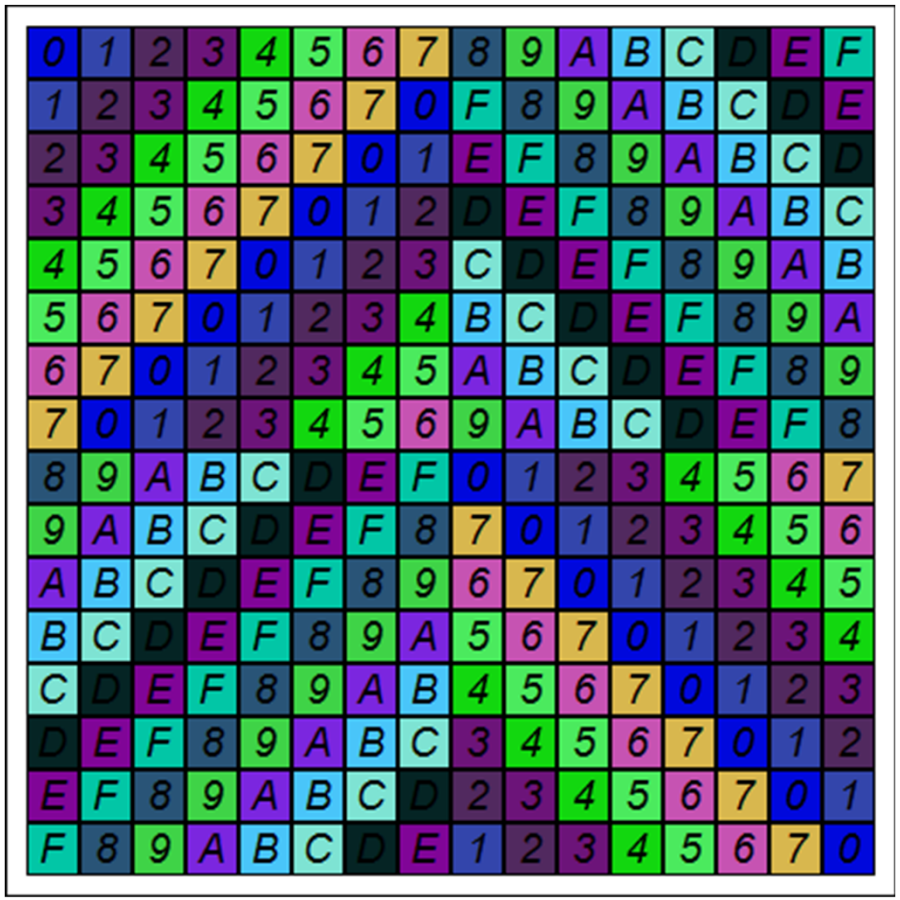






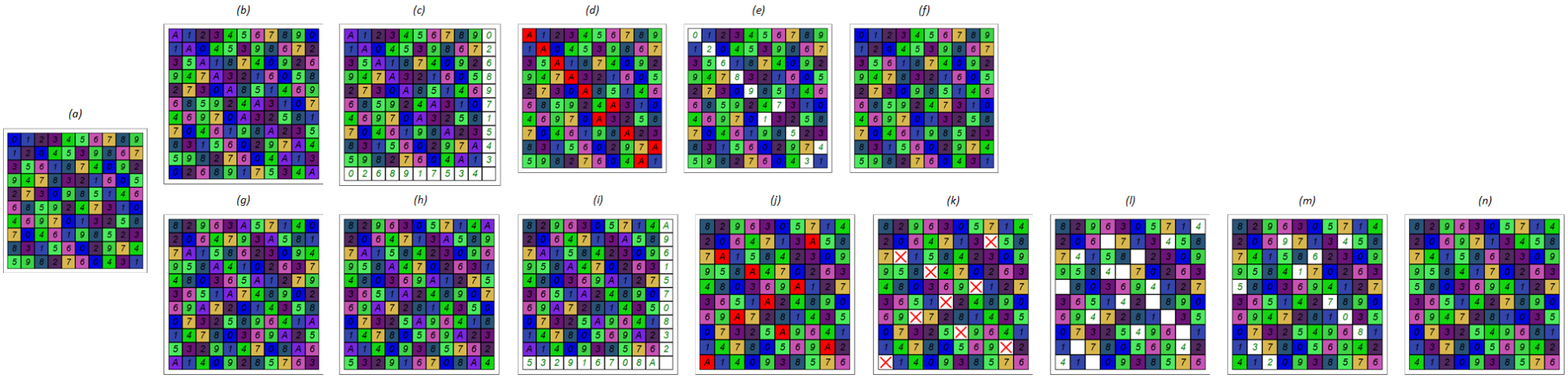
Some new theoretical results: pine LS

- upper and lower bounds for maximal number of intercalates a(n) in LS
  - **a(n) <= (n*(n-1))^2 in theory, a(n) << (n*(n-1))^2 in practice, a(n) < O(n^4) asymptotically! Polynomial, not exponential grows!**
  - **o(k1*n^3) < a(n), k1=1/8, for orders N=2*k due to pine LS existence!**



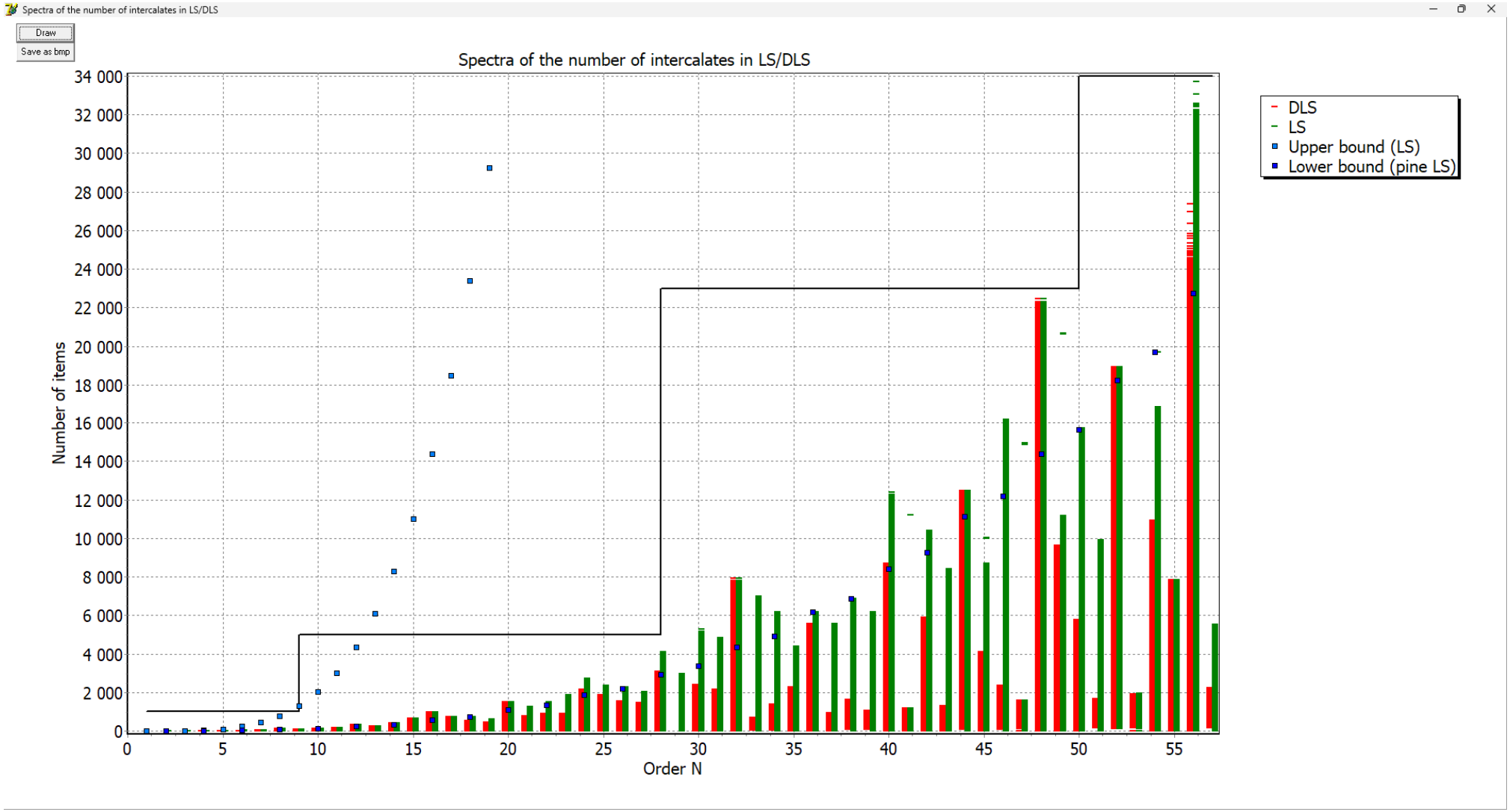
LS of order N+1 to LS of order N transformation

From LS N+1 to LS N transformation

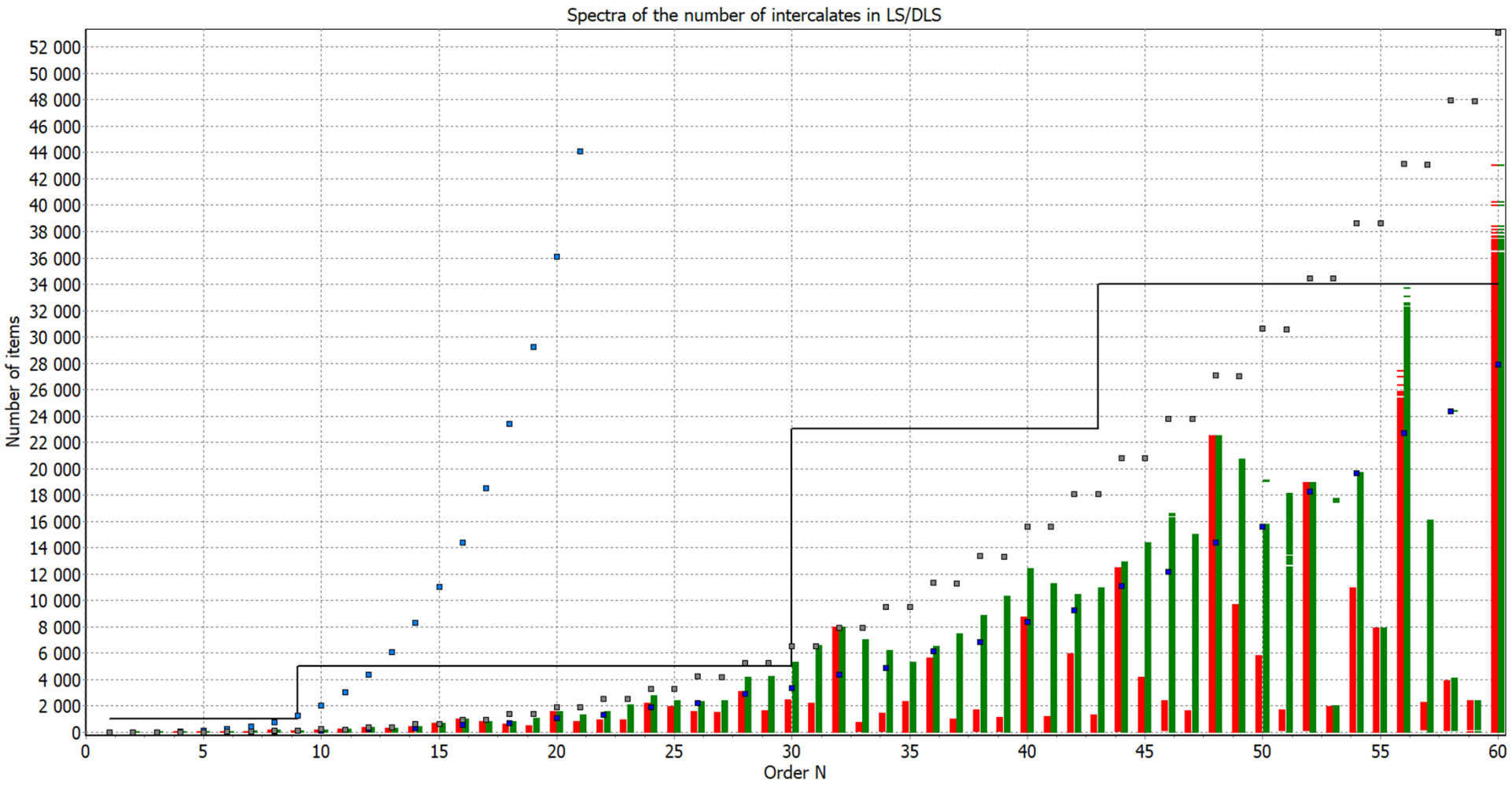


Experimental results

Current experiments state (IIS'26, Kursk, September)



Current experiments state (NSCF'26, Pereslavl'-Zalessky, November)



New results (NSCF'21 → NSCF'22 → NSCF'23 → NSCF'24 → GRID'25 → Recogn'25 → IIS'26 → NSCF'26)

Cardinalities of the heuristic approximations of the **DLS** spectra:

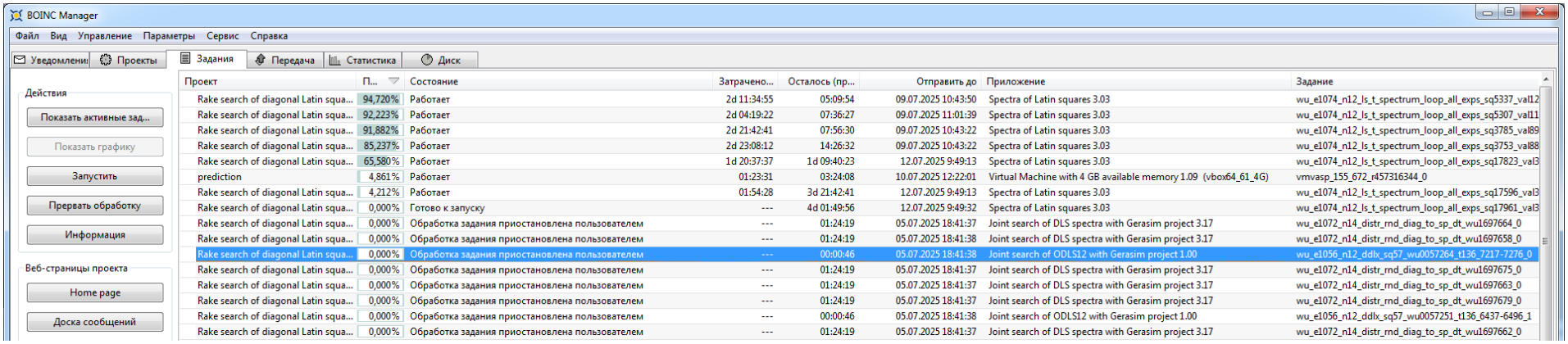
- intercalates in **DLS** (https://oeis.org/A345760):
  - a(9)≥62 → a(9)=64 (2022)
  - ? → a(10)≥88 → a(10)≥93 → a(10)≥98 (2024)
  - ? → a(11)≥100 → a(11)≥101 → a(11)≥145 (2024)
  - ? → a(12)≥210 → a(12)≥210 → a(12)≥259 (2024)
  - ? → a(13)≥152 → a(13)≥152 → a(13)≥197 → **a(13)≥246** **(2026)**
  - ? → ? → a(14)≥337 → a(14)≥362 (2024)
  - ? → ? → a(15)≥18 → a(15)≥536 (2024)
  - ? → ? → a(16)≥11 → a(16)≥789 → a(16)≥792 (2025)
  - ? → ? → a(17)≥7 → a(17)≥685 → **a(17)≥705** **(2026)**
  - ? → ? → ? → a(18)≥419 → a(18)≥535 → **a(18)≥538** **(2026)**
  - ? → ? → a(19)≥15 → a(19)≥447 → **a(19)≥452** **(2026)**
  - ? → ? → ? → a(20)≥1009 → a(20)≥1362 (2025)
  - ? → ? → ? → a(21)≥740 → a(21)≥772 (2025)
  - ? → ? → ? → a(22)≥737 → a(22)≥872 (2025)
  - ? → ? → ? → a(23)≥885 → a(23)≥887 (2025)
  - ? → ? → ? → a(24)≥1610 → a(24)≥2034 → **a(24)≥2036** **(2026)**
  - ? → ? → ? → ? → a(25)≥1856 (2025)
  - ? → ? → ? → ? → a(26)≥1477 → **a(26)≥1508** **(2026)**
  - ? → ? → ? → ? → a(27)≥1337 → **a(27)≥1435** **(2026)**
  - ? → ? → ? → ? → a(28)≥2829 → **a(28)≥3015** **(2025)**
  - ? → ? → ? → ? → ? → ? → **a(29)≥1552** **(2026)**
  - ? → ? → ? → ? → ? → ? → **a(30)≥2362 in process…** **(2026)**



Additional information

Current experiments set

- spectrum of ODLS number for DLS of order 12;
  - low part – 1 threaded software implementation, **active** (RakeSearch project in future)
  - high part – RakeSearch project (experiment number e1056, **active**)
- filling the collection of ODLS of order 10 – RakeSearch project (experiment number e43, **active**)
- distributed random diagonalizing of spectrum of diagonal transversals number of DLS of order 14 – RakeSearch project (experiment number e1072, **active**)
- expanding spectra of intercalates number in LS/DLS of orders N>9 RakeSearch project (experiments numbers e1076, e1078, **active**)



RakeSearch volunteer distributed computing project

- real performance – **10-15 TFLOP/s** (by benchmark information), **7 TFLOP/s** (by http://boincstats.com information), maybe more…

- users number – more than **2400** (269 now is active)
- hosts number – more than **12500** (709 now is active)
- teams number – **194** (63 now is active)

- https://rake.boincfast.ru/rakesearch/



Spectra of small order: Brute Force approach

DLS order	*1*	*4*	*5*	*6*	*7*	*8*	*9*
Transversals	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*
Diagonal transversals	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*
Intercalates	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*
ODLS	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*

- computing experiment for order N=9 required 5 months of calculations using volunteer computing projects (Gerasim@Home + RakeSearch) with real performance 7—8 TFLOP/s and a set of parallel worked subprojects



Spectrum expanding methods

Neighborhoods analyzing (abbr. **M1**)

- spectrum closure, parallelized by different squares in spectrum

Square by square neighborhoods analyzing (abbr. **M2**)

- separate closure of spectra starting from the squares of the original spectrum, only for small orders (N<13)

Diagonalizing (abbr. **d**)

- full (abbr. **fd**, for all squares in spectrum), only for small orders (N<14)
- partial (heuristic, abbr. **hd**) – some first experiments, for order N=14 (now) and higher (in future)

Methods based on movement in space of squares (in future for orders N>15-16, constructing full spectra is difficult)

- greedy movement «down» and «up»
- random walks
- weighted random walks
- other discrete combinatorial optimization methods



More detailed representation of the experiments

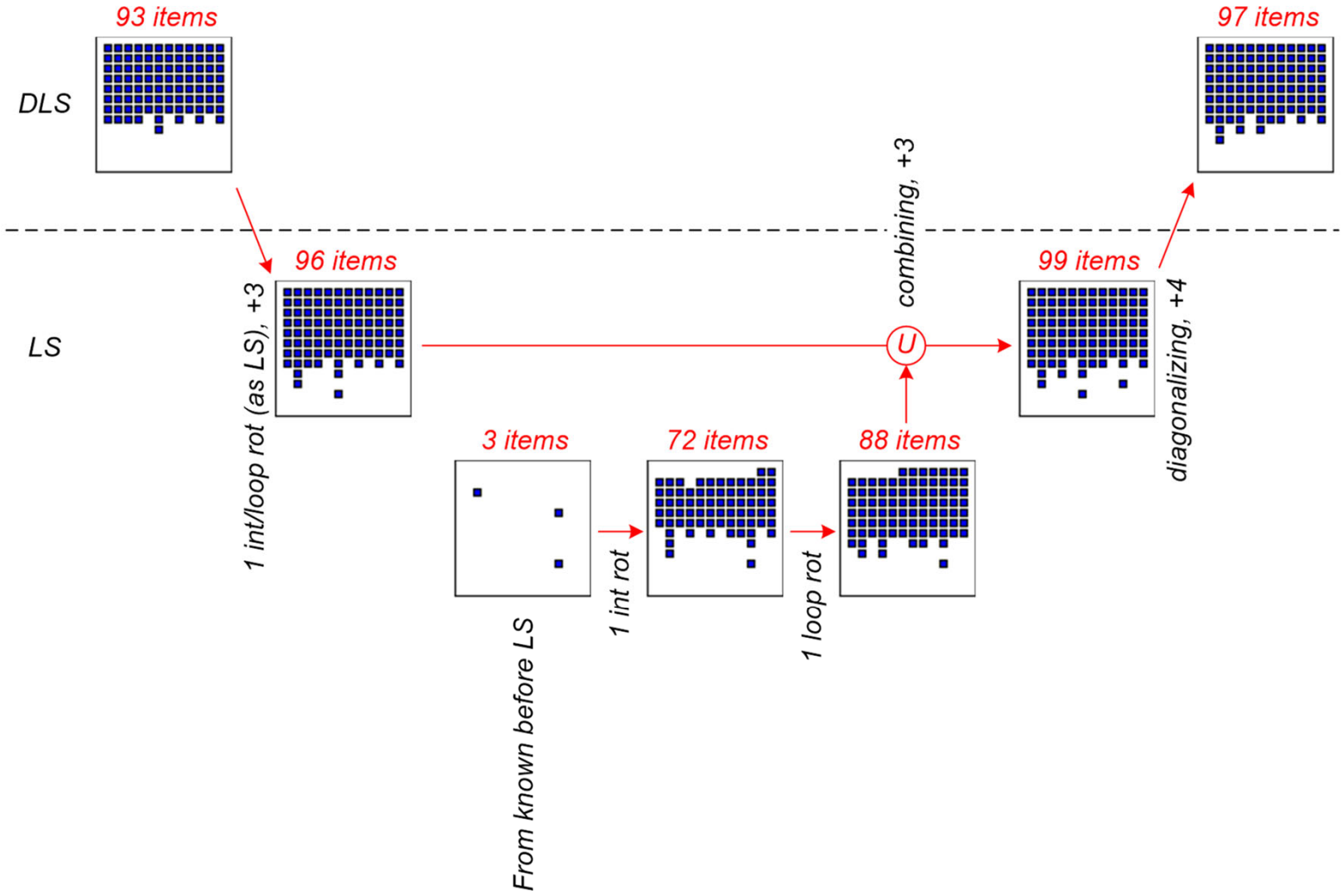
DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M1	To Do	To Do	x
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M1	M1	x
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	To Do	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s	To Do	x?	x?	



From DLS to LS and back to DLS

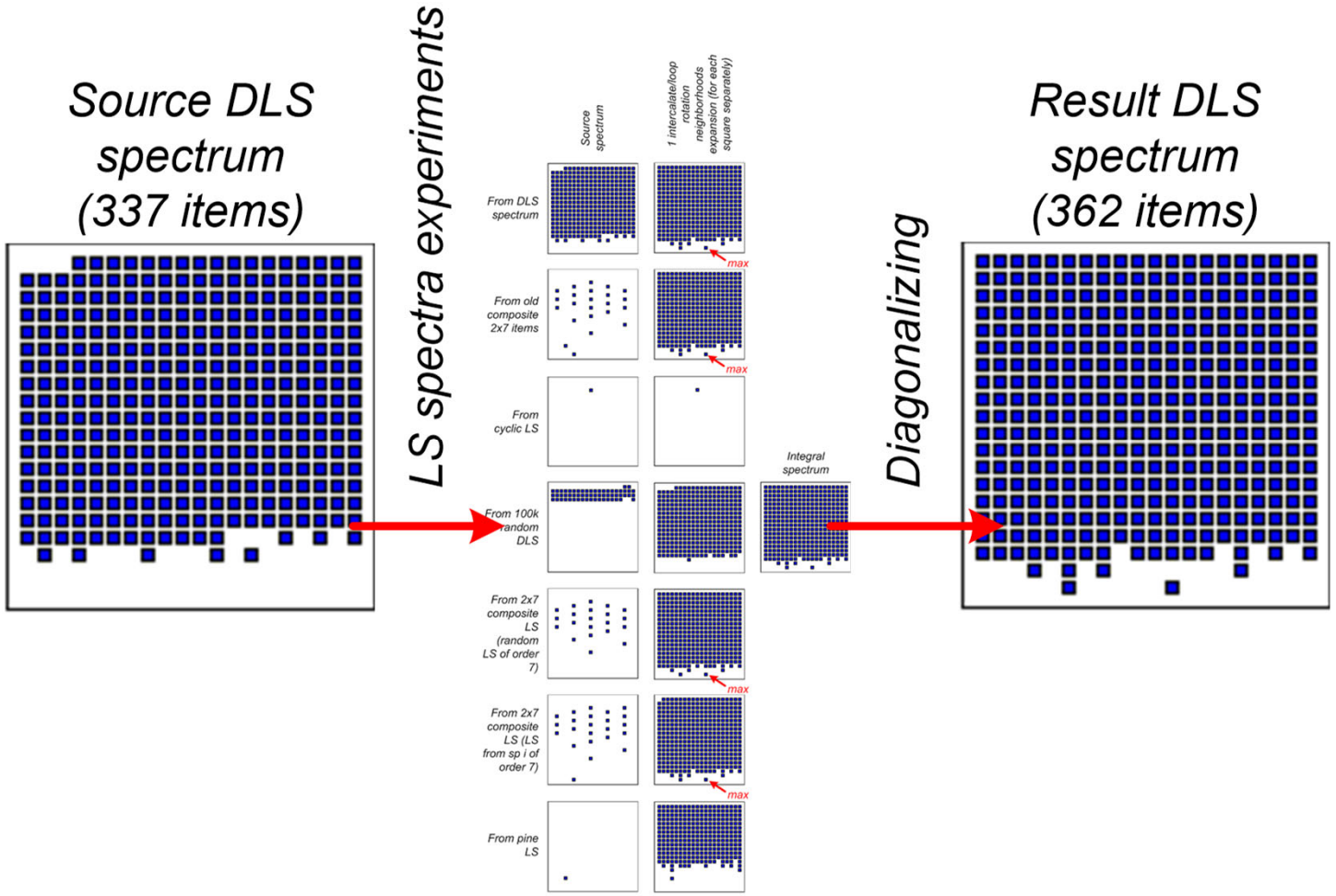
- base DLS spectrum
- expanding DLS spectrum (working with DLS as LS) – resulting **LS** spectrum
- diagonalizing LS from LS spectrum – resulting **DLS** spectrum

Order N=10 example



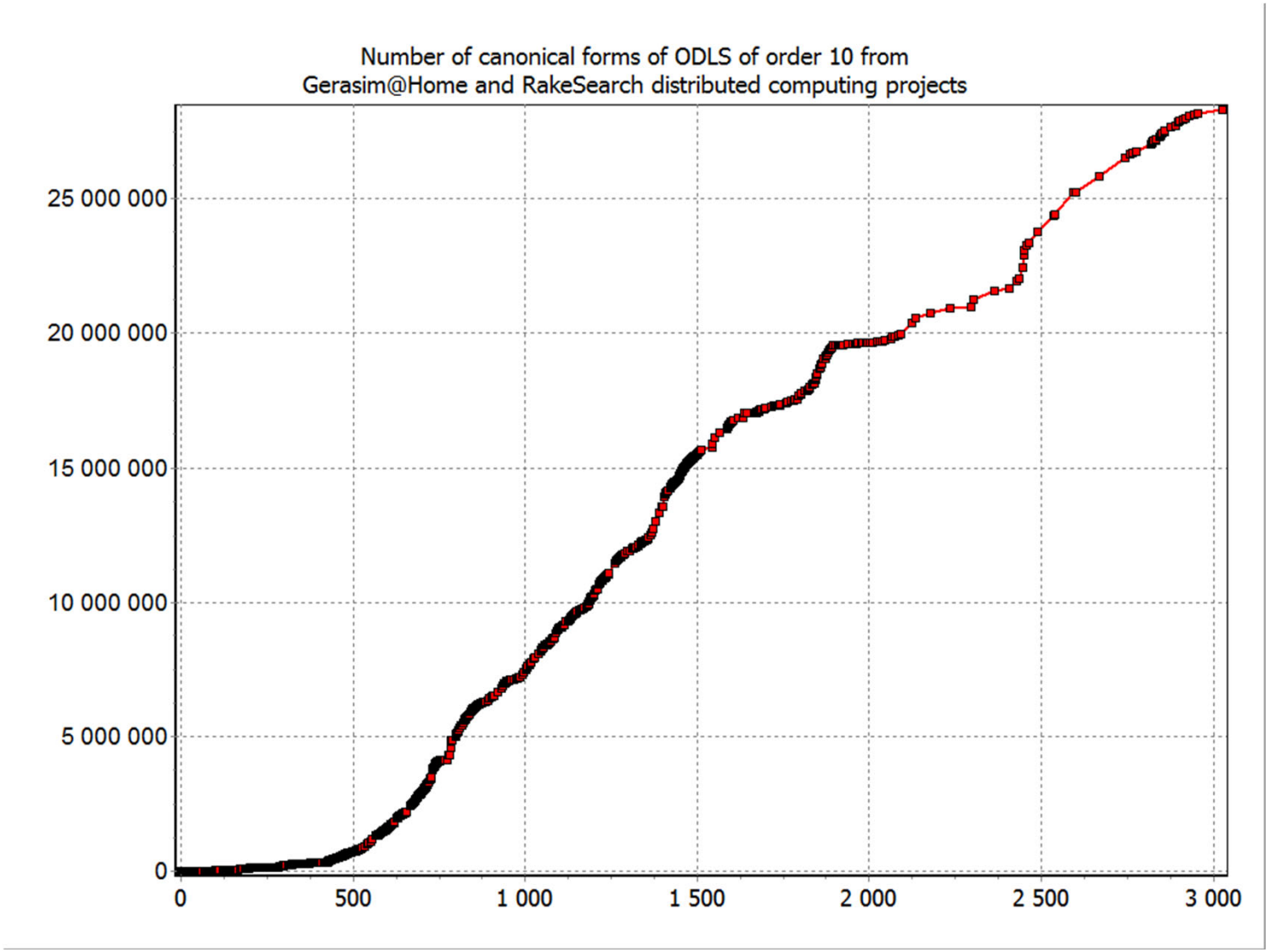
From LS to DLS spectrum using diagonalization example for order N=14

- base DLS spectrum
- expanding DLS spectrum (working with DLS as LS) – resulting **LS** spectrum
- diagonalizing LS from LS spectrum – resulting **DLS** spectrum



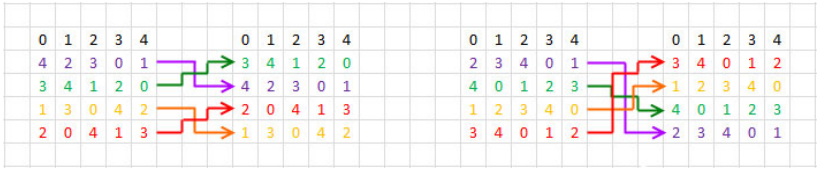
- difficult (impossible?) from orders N≥16
- randomization of diagonalizing procedure?

Searching for ODLS of order 10 experiment results



For future:

- combination of methods for searching for ODLS and heuristic approximation of spectra;
- extended parastrophic transformations;
- …



Thank you for your attention!

Thanks to all the volunteers who take part in the RakeSearch volunteer distributed computing project!

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