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LIST OF THE COMBINATORIAL STRUCTURES FROM DLSS OF ORDERS 1–8 ON ORTHOGONALITY RELATIONSHIP SET

20.02.2019

Basic definitions

DLS – diagonal Latin square of order *N*.

Orthogonal DLSs (ODLS) – pair of DLS *A* and *B*, in which all ordered pairs of elements $\left(a_{ij}, b_{ij}\right)$ are distinct.

String representation of DLS – elements of DLS that are written from left to right from top to bottom:

3 2 8 4 6 7 1 0 9 5
8 1 2 7 4 6 3 5 0 9
1 5 0 9 8 2 4 3 7 6
6 8 5 2 0 9 7 1 4 3
9 0 7 1 5 4 2 6 3 8
4 3 9 0 1 8 6 7 5 2
0 6 3 8 7 5 9 2 1 4
5 7 6 3 9 1 8 4 2 0
7 9 4 5 2 3 0 8 6 1
2 4 1 6 3 0 5 9 8 7

3284671095812746350915098243766852097143907154263843901867520638759214576391842079452308612416305987

DLX – Dancing Links X algorithm – algorithm for exact cover problem solving [3].

Canonical form (CF) – lexicographically minimal string representation of DLS within corresponding isomorphism (isotopism) class [4].

Combinatorial structure – graph, in which vertices are DLSs and edges are corresponds to the orthogonality binary relationship between pair of DLSs. The set of different combinatorial structures that are found within distributed computing project Gerasim@Home is shown below.

1. COMBINATORIAL STRUCTURES FROM DLS OF ORDER *N*=1

1.1. Once

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:
DLS 1: 0 (DLS *A*, CF 1);
DLS 2: 0 (DLS *B*, CF 2) – DLS 1 and 2 are same.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:
CF 1: 0 (DLS *A*, DLS *B*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 2, a = 1, \rho = [1, 1].$$

Method of finding:
trivial case, square is orthogonal by itself (reflexivity property of orthogonality binary relationship, valid only for order *N*=1).

Different CFs set for order *N*=1:
ONCE (A):1 – 1, where:
1 CFs – 1

2. COMBINATORIAL STRUCTURES FROM DLS OF ORDER *N*=2 AND *N*=3

For orders *N*=2 and *N*=3 DLSs are not exist.

3. COMBINATORIAL STRUCTURES FROM DLS OF ORDER *N*=4

3.1. Once

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123230132101032 (DLS *A*, CF 1);
DLS 2: 0123321010322301 (DLS *B*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123230132101032 (DLS *A*, DLS *B*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 2, a = 1, \rho = [1, 1]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

Different CFs set for order *N*=4:

ONCE (A):1 – 1, where:
1 CFs – 1

4. COMBINATORIAL STRUCTURES FROM DLS OF ORDER *N*=5

4.1. DLS without ODLS (bachelor)

a) A *b)* 1

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123413420421032034134012 (DLS *A*, CF 1).

Adjacency matrix:

$$M = (0).$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123413420421032034134012 (DLS *A*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 1, a = 0, \rho = [1].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

4.2. Once

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123423401401231234034012 (DLS *A*, CF 1);
<u>DLS 2:</u> 0123434012123404012323401 (DLS *B*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123423401401231234034012 (DLS *A*, DLS *B*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 2,\ a = 1,\ \rho = [1, 1].$

Method of finding:

Brute Force + Euler-Parker method + DLX.

Different CFs set for order *N*=5:

ONCE (A):1 – 1, where:
1 CFs – 1

5. COMBINATORIAL STRUCTURES FROM DLS OF ORDER *N*=6

5.1. DLS without ODLS (bachelor)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 012345120534435021301452543210254103 (DLS *A*, CF 1).

Adjacency matrix:

$$M = (0).$$

Different CFs set within combinatorial structure:

CF 1: 012345120534435021301452543210254103 (DLS *A*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 1, a = 0, \rho = [1].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

Remark:

for order *N*=6 ODLSs are not exist.

6. COMBINATORIAL STRUCTURES FROM DLS OF ORDER *N*=7

6.1. DLS without ODLS (bachelor)

a) $\boxed{A}$ b) $\boxed{1}$

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456120653454310623654201401562365423102360145 (DLS *A*, CF 1).

Adjacency matrix:

$$M = (0).$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456120653454310623654201401562365423102360145 (DLS *A*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 1, a = 0, \rho = [1].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

6.2. Once

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456124563064520135364102461032535012642036541 (DLS *A*, CF 1);
DLS 2: 0123456630124535641204652031143650220456135210364 (DLS *B*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456124563064520135364102461032535012642036541 (DLS *A*, DLS *B*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 2, a = 1, \rho = [1, 1].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

6.3. Clique-4

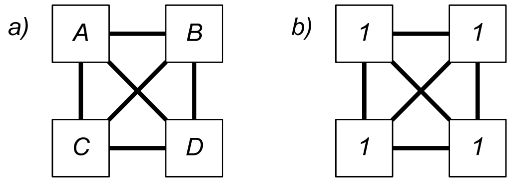

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456231564056401234062315620153415340623456201 (DLS *A*, CF 1);
<u>DLS 2:</u> 0123456620153434562011534062231564040623155640123 (DLS *B*, CF 1);
<u>DLS 3:</u> 0123456564012315340626201534345620123156404062315 (DLS *C*, CF 1);
<u>DLS 4:</u> 0123456345620140623152315640564012362015341534062 (DLS *D*, CF 1);

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456231564056401234062315620153415340623456201 (DLS *A*, DLS *B*, DLS *C*, DLS *D*);

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 4$, $a = 6$, $\rho = [4, 4, 4, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

6.4. Different ODLS of order 7 CFs set with corresponding combinatorial structures

ONCE (A):1 – 4, where:
1 CFs – 4

CLIQUE4 (O):3 – 1, where:
1 CFs – 1

7. COMBINATORIAL STRUCTURES FROM DLS OF ORDER *N*=8

7.1. DLS without ODLS (bachelor)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712045736673014524675210324567310351206747061324553476021 (DLS *A*, CF 1).

Adjacency matrix:

$M = (0)$.

Different CFs set within combinatorial structure:

CF 1: 0123456712045736673014524675210324567310351206747061324553476021 (DLS *A*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 1, a = 0, \rho = [1]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.2. Asymmetric once

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712047635537160427635120464523170306754212546071347102356 (DLS A, CF 1);

DLS 2: 0123456776510243673051242314607540752316540617321267345035427601 (DLS B, CF 2).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712047635537160427635120464523170306754212546071347102356 (DLS A);

CF 2: 0123456712307456374256106501374273546201561703244065217324761035 (DLS B).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 2,\ a = 1,\ \rho = [1, 1].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.3. Symmetric once

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712357604371024567654321053716042406753216542017324061735
(DLS *A*, CF 1) – central symmetry;

DLS 2: 0123456757012436756431201345607264321705365072142076534142170653
(DLS *B*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712357604371024567654321053716042406753216542017324061735
(DLS *A*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 2, a = 1, \rho = [1, 1]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.4. Line-3 without symmetry (asymmetric twice)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456712047635537160427635120464523170376054212546071340172356 (DLS *A*, CF 1);

<u>DLS 2:</u> 0123456723451706765432106071543257362041421706531402637535607124 (DLS *B*, CF 2);

<u>DLS 3:</u> 0123456716047235537120467235160464523170376054212546071340176352 (DLS *C*, CF 3).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456712047635537160427635120464523170376054212546071340172356 (DLS *A*);

<u>CF 2:</u> 0123456712375604406573215401673276543210371204566540217323761045 (DLS *B*);

<u>CF 3:</u> 0123456712306745567410327456210363415270406273512507341637150624 (DLS *C*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 3, a = 2, \rho = [1, 1, 2]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.5. Line-3 with symmetry (symmetric twice)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456712547630306724517645120354123076230167454576031267305124
(DLS *A*, CF 1) – central symmetry;

<u>DLS 2:</u> 0123456724306751631572401576302476512403576403123042517642071635
(DLS *B*, CF 2) – central symmetry;

<u>DLS 3:</u> 0123456776510423547610324032671567153240324751062304765115602374
(DLS *C*, CF 1) – central symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456712547630306724517645120354123076230167454576031267305124
(DLS *A*, DLS *C*);

<u>CF 2:</u> 0123456712307456506723413674120543125670250167347456301267450123
(DLS *B*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 3, a = 2, \rho = [1, 1, 2]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.6. Asymmetric line-4

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712370456354276015671324073546012671053244065217324061735 (DLS B , CF 1) – vertical symmetry;

DLS 2: 0123456756047321147520367352610437615240203614756540371242170653 (DLS A , CF 2) – vertical symmetry;

DLS 3: 0123456725167034426701537054231616453270530167423470162567325401 (DLS C , CF 3) – vertical symmetry;

DLS 4: 0123456767425301743026151506327430541726261574304267015353716042 (DLS D, CF 4) – vertical symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712370456354276015671324073546012671053244065217324061735 (DLS B);

CF 2: 0123456712370456247160353605271457643102451076237356124060425371 (DLS A);

CF 3: 0123456712370456356071247654321053061742674523014012567324716035 (DLS C);

CF 4: 0123456712043756365072144567012324716035574613027312564060352471 (DLS D).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 4, a = 3, \rho = [1, 1, 2, 2]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.7. Loop-4 (1 different CF)

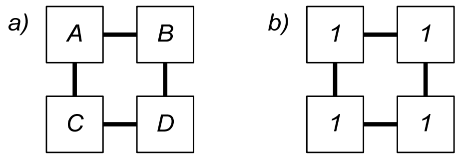

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712547036306724514675120354123670230167457546031267305124
(DLS *A*, CF 1) – central symmetry;

DLS 2: 0123456724706153631572407546302136512704576403121032547642071635
(DLS *B*, CF 1) – central symmetry;

DLS 3: 0123456767452301263401757501342630765214145276305260174343176052
(DLS *C*, CF 1) – central symmetry;

DLS 4: 0123456776310425547610324052671327153640324751066304725115602374
(DLS *D*, CF 1) – central symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712547036306724514675120354123670230167457546031267305124
(DLS *A*, DLS *B*, DLS *C*, DLS *D*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 4,\, a = 4,\, \rho = [2, 2, 2, 2]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

Remark:

loop-4 structure also can be treated as rhombus-2.

7.8. Triple (structure 1:3)

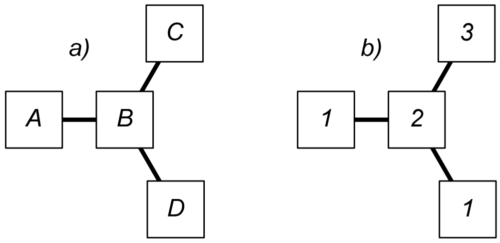

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456712306745645701237546301246751230576423013012547623017654
(DLS *A*, CF 1) – vertical symmetry;

<u>DLS 2:</u> 0123456774563021103276544675123057642103230167456547031232105476
(DLS *B*, CF 2) – vertical symmetry, column-inverse DLS;

<u>DLS 3:</u> 0123456746750312721654305764302113027645345012762031675465472103
(DLS *C*, CF 3) – vertical symmetry;

<u>DLS 4:</u> 0123456756742130674512032301567432107456456703127456302110326745
(DLS *D*, CF 1) – vertical symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456712306745645701237546301246751230576423013012547623017654
(DLS *A*, DLS *D*);

<u>CF 2:</u> 0123456712305746576401324675320175461023645723103012647523017654
(DLS *B*);

<u>CF 3:</u> 0123456712370456604523715371604245607123371256047456123026043715
(DLS *C*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 4,\ a = 3,\ \rho = [1, 1, 1, 3]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

Remark:

unlike order N=10, all triples for order N=8 includes only 3 CFs.

7.9. Symmetric four (structure 1:4), 3 CF

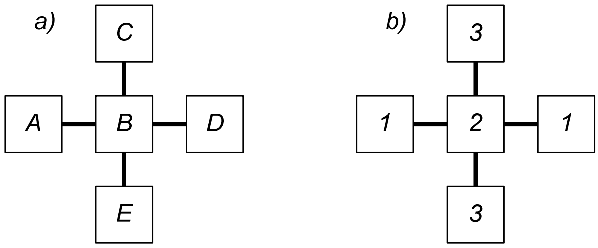

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456712370456371526046574302153016742764253104056127324607135 (DLS *A*, CF 1) – horizontal symmetry;

<u>DLS 2:</u> 0123456774561230103254765647031247652103231076453201675465743021 (DLS *B*, CF 2) – horizontal symmetry;

<u>DLS 3:</u> 0123456712307456371526046504372153716042764253104056127324670135 (DLS *C*, CF 3) – horizontal symmetry;

<u>DLS 4:</u> 0123456715307426371526046204375123716045764253104056127354670132 (DLS *D*, CF 1) – horizontal symmetry;

<u>DLS 5:</u> 0123456715370426371526046274305123016745764253104056127354607132 (DLS *E*, CF 3) – horizontal symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456712370456371526046574302153016742764253104056127324607135 (DLS *A*, DLS *D*);

<u>CF 2:</u> 0123456712307456367250147456123050143672456701236305274127416305 (DLS *B*);

<u>CF 3:</u> 0123456712307456371526046504372153716042764253104056127324670135 (DLS *C*, DLS *E*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 5, a = 4, \rho = [1, 1, 1, 1, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.10. Symmetric four (structure 1:4), 4 CF

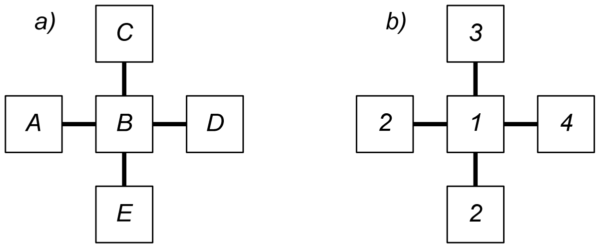

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456712305746756401236475321057461032465723013012647523017654
(DLS *B*, CF 1) – vertical symmetry;

<u>DLS 2:</u> 0123456753617024167523403042517642306715751604322407165367543201
(DLS *A*, CF 2);

<u>DLS 3:</u> 0123456753617024127563403046517246302715751204362407165367543201
(DLS *C*, CF 3) – vertical symmetry;

<u>DLS 4:</u> 0123456753617024167023453542017642356710701654322407165367543201
(DLS *D*, CF 4) – vertical symmetry;

<u>DLS 5:</u> 0123456753617024127063453546017246352710701254362407165367543201
(DLS *E*, CF 2).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456712305746756401236475321057461032465723013012647523017654
(DLS *B*);

<u>CF 2:</u> 0123456712370654301524767456123065407123476253015371604226043715
(DLS *A*, DLS *E*);

<u>CF 3:</u> 0123456712370456371256045476103245607123604523717351624026043715
(DLS *C*);

<u>CF 4:</u> 0123456712370456301526747456123045607123674253015371604226043715
(DLS *D*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 5, a = 4, \rho = [1, 1, 1, 1, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.11. Asymmetric four (structure 1:4), 5 CF

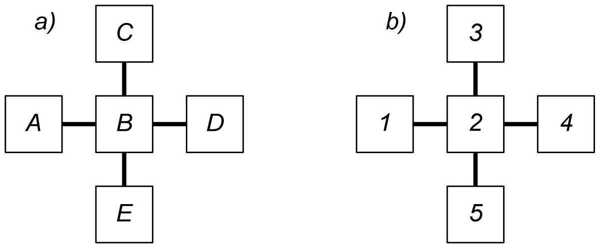

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712057634763412054567012353716042375624102410375660425371 (DLS *A*, CF 1);

DLS 2: 0123456743726015126734503715260465407321203451767651024354061732 (DLS *B*, CF 2);

DLS 3: 0123456717052634763412054562017353716042325674102410375660475321 (DLS *C*, CF 3);

DLS 4: 0123456762517430751602435407632146753102374026152034175613625074 (DLS *D*, CF 4);

DLS 5: 0123456767512430751602435402637146753102324076152034175613675024 (DLS *E*, CF 5).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712057634763412054567012353716042375624102410375660425371 (DLS *A*);

CF 2: 0123456712307456354706126751324053046721761253044065217324761035 (DLS *B*);

CF 3: 0123456712367405507163427405123646523170374056212564071363172054 (DLS *C*);

CF 4: 0123456712057643354710264376510250613274643207517654231027106435 (DLS *D*);

CF 5: 0123456712307456645123702574061373026145301657245647103247653201 (DLS *E*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 5, a = 4, \rho = [1, 1, 1, 1, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.12. Rhombus-4 (3 CF)

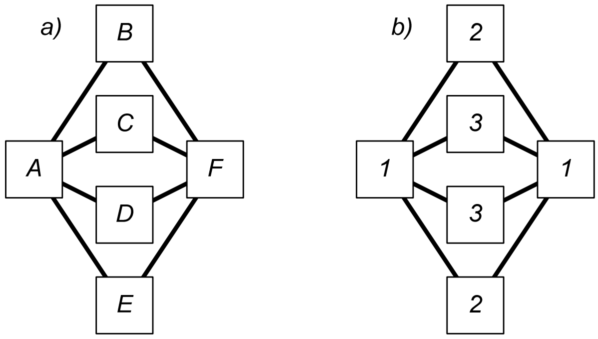

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712307654564720316475130230126745756431202301547647560213 (DLS *A*, CF 1);

DLS 2: 0123456776452031457602132301674567543120321054761032765454671302 (DLS *B*, CF 2);

DLS 3: 0123456747652031745602132301547656743120321076541032674565471302 (DLS *C*, CF 3);

DLS 4: 0123456776450213457613021032674567543120230154763210765454672031 (DLS *D*, CF 3);

DLS 5: 0123456747650213745613021032547656743120230176543210674565472031 (DLS *E*, CF 2);

DLS 6: 0123456764571302231076545746203132015476103267457564312046750213 (DLS *F*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712307654564720316475130230126745756431202301547647560213 (DLS *A*, DLS *F*);

CF 2: 0123456723016745476503216547210332165074765214305074321614307652 (DLS *B*, DLS *E*);

CF 3: 0123456723016745476503216547210334105672705412365672341012367054 (DLS *C*, DLS *D*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 6, a = 8, \rho = [2, 2, 2, 2, 4, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.13. Hedgehog-3 (6 CF)

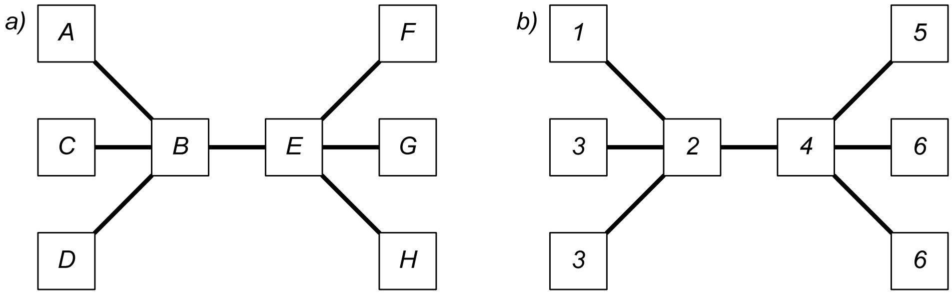

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456723017654156732407452603152741306671504234036517236402715 (DLS *A*, CF 1) – vertical symmetry;

<u>DLS 2:</u> 0123456776102345624150734365170234567120503762141572043627043651 (DLS *B*, CF 2) – vertical symmetry;

<u>DLS 3:</u> 0123456725017436146732507632504152741603471603253045617263502714 (DLS *C*, CF 3);

<u>DLS 4:</u> 0123456713720654756031422457603150142376670514234236571036417205 (DLS *D*, CF 3);

<u>DLS 5:</u> 0123456715720436746031522637504150142673470613253245671063517204 (DLS *E*, CF 4) – vertical symmetry;

<u>DLS 6:</u> 0123456723017654671504235274130674526031156732404036517236402715 (DLS *F*, CF 5) – vertical symmetry;

<u>DLS 7:</u> 0123456776012354621504735364170234567021153762404072513627403615 (DLS *G*, CF 6);

<u>DLS 8:</u> 0123456723107645674150234275130674526130506732141536047236042751 (DLS *H*, CF 6).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456723017654156732407452603152741306671504234036517236402715 (DLS *A*);

<u>CF 2:</u> 0123456712370456247160357365214067543201401256735640731235061724 (DLS *B*);

<u>CF 3:</u> 0123456712346705257016434716235063015274705234165467013236457021 (DLS *C*, DLS *D*);

<u>CF 4:</u> 0123456712043756237610454715260330516274763254105467013265407321 (DLS *E*);

CF 5: 0123456723017654176502435472630172541036651734204036517236402715
(DLS *F*);
CF 6: 0123456712305746365712042764301574056132451673205041267363720451
(DLS *H*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 8, a = 7, \rho = [1, 1, 1, 1, 1, 1, 4, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

Remark:

For different orders of DLS there are different types of hedgehogs (for example, hedgehog-6 for order *N*=9).

7.14. Hedgehog-3 (8 CF)

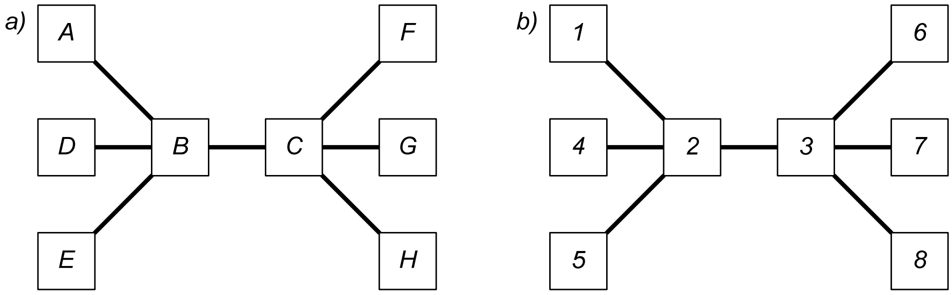

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456753607142465702132714360532716054140257367536142060452371
(DLS *A*, CF 1) – horizontal symmetry;

DLS 2: 0123456712043756237160453056127474625130673524015647031245107623
(DLS *B*, CF 2) – horizontal symmetry;

DLS 3: 0123456763507241451706232764310536752014540167327236145010425376
(DLS *C*, CF 3) – horizontal symmetry;

DLS 4: 0123456764507231751436202367014536052714504163724276105317325406
(DLS *D*, CF 4) – horizontal symmetry;

DLS 5: 0123456754607132765432102317064532016754104253764576102367352401
(DLS *E*, CF 5) – horizontal symmetry, string-inverse square;

DLS 6: 0123456732016754267430154056127374125630174523065367014265307421
(DLS *F*, CF 6) – horizontal symmetry;

DLS 7: 0123456712743056735162403506172454607132603254712647031547152603
(DLS *G*, CF 7) – horizontal symmetry;

DLS 8: 0123456732716054765432104506172354107632104253762367014567352401
(DLS *H*, CF 8) – horizontal symmetry, string-inverse square.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712043756371526044657021354716032654073217362514020361475
(DLS *A*);

CF 2: 0123456712043756237160453056127474625130673524015647031245107623
(DLS *B*);

CF 3: 0123456712043756375162045376104245107623263704157462513060452371
(DLS *C*);

CF 4: 0123456712307456475612037514362053016742264703153062517464752031
(DLS *D*);

CF 5: 0123456712307456471526037654321053016742654703213062517424761035 (DLS *E*);

CF 6: 0123456712043756357160247356124047125603263704155460713260452371 (DLS *F*);

CF 7: 0123456712043756357610244065217373516240563074122417063567425301 (DLS *G*);

CF 8: 0123456712307456371256047654321054716032654703214065217323061745 (DLS *H*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 8, a = 7, \rho = [1, 1, 1, 1, 1, 1, 4, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.15. Dandelion

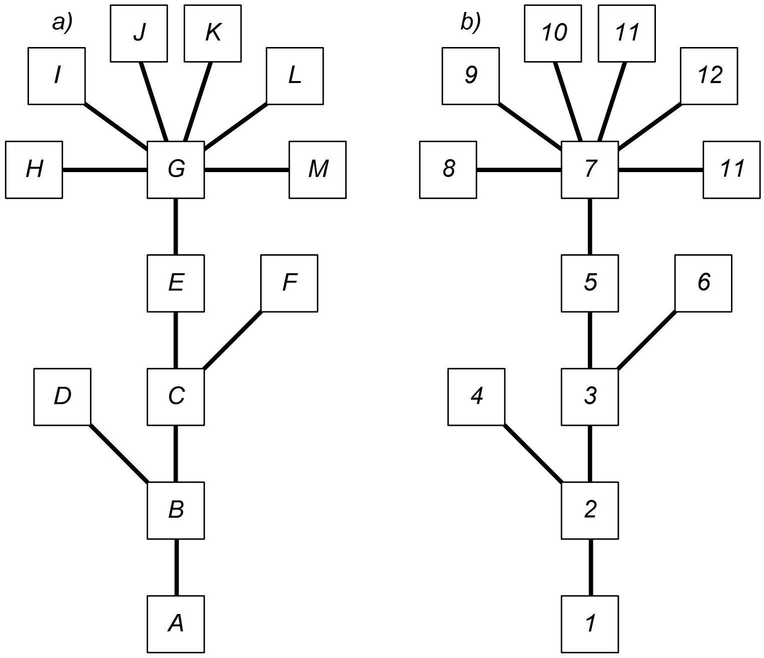

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456723016745145702367246135067352401456071235014367236725014 (DLS *A*, CF 1) – horizontal symmetry;

DLS 2: 0123456776543210654073213062517454761032230167451237045647152603 (DLS *B*, CF 2) – horizontal symmetry, string-inverse square;

DLS 3: 0123456767452301421706535306174276325410105432762471603535607124 (DLS *C*, CF 3) – horizontal symmetry;

DLS 4: 0123456723016745427610531754320660325471351076247645231054670132 (DLS *D*, CF 4) – horizontal symmetry;

DLS 5: 0123456764752031536701427231645046507213104253763506172427143605 (DLS *E*, CF 5) – horizontal symmetry;

DLS 6: 0123456745670123763254103276105417452306240167355310764260543271 (DLS *F*, CF 6) – horizontal symmetry;

DLS 7: 0123456732107654105432762367014554716032673524017642531045061723 (DLS *G*, CF 7) – horizontal symmetry;

DLS 8: 0123456764752031531706427236145046507213104253763501672427643105 (DLS *H*, CF 8) – horizontal symmetry;

DLS 9: 0123456764752031231706457536142046507213104253763201675457643102 (DLS *I*, CF 9) – horizontal symmetry;

DLS 10: 0123456764752031236701457531642046507213104253763206175457143602 (DLS *J*, CF 10) – horizontal symmetry;

DLS 11: 0123456724756031536701427631245042507613104253763506172467143205 (DLS *K*, CF 11);

DLS 12: 0123456724716035536701427635241042107653104253763506172467543201 (DLS *L*, CF 12) – horizontal symmetry;

DLS 13: 0123456764712035536701427235641046107253104253763506172427543601 (DLS *M*, CF 11).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456723016745145702367246135067352401456071235014367236725014 (DLS A);

<u>CF 2:</u> 0123456712306745645723104576102376453201576401323012547623017654 (DLS B);

<u>CF 3:</u> 0123456723016745174253066475203135607124421706537654321050361472 (DLS C);

<u>CF 4:</u> 0123456723016745357610246054327117325406421076537645231054670132 (DLS D);

<u>CF 5:</u> 0123456712043756273164056475203153607142451706237642531030561274 (DLS E);

<u>CF 6:</u> 0123456712307456306251744517062374561230674523015301674226743015 (DLS F);

<u>CF 7:</u> 0123456723016745327610541054327667425301451706237635241054607132 (DLS G);

<u>CF 8:</u> 0123456712307456476251037514362030561274630527415471603226470315 (DLS H);

<u>CF 9:</u> 0123456712346750537021464761320574165023354206716057143226057314 (DLS I);

<u>CF 10:</u> 0123456712057436267031547514062357326041405613723467521063412705 (DLS J);

<u>CF 11:</u> 0123456712346705371504267056214345017632246053716347125056723014 (DLS K, DLS M);

<u>CF 12:</u> 0123456712370456476521037451623035107624604253715376104226043715 (DLS L).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 13,\ a = 12,\ \rho = [1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 7]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.16. Cactus

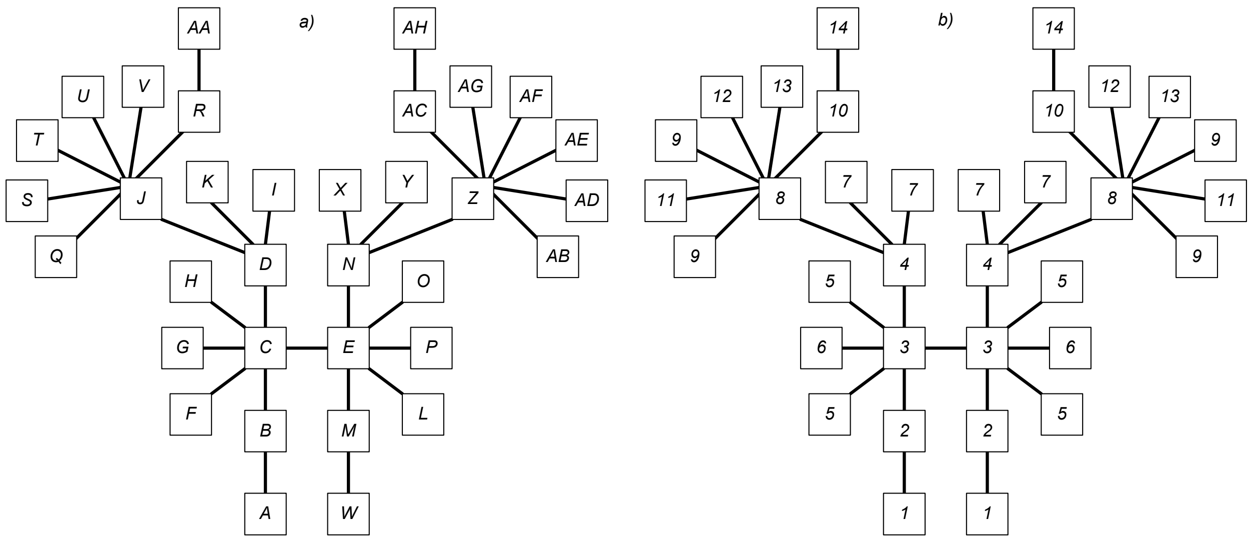

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712307456406521732714360576425310350617245471603263570241 (DLS *A*, CF 1) – horizontal symmetry;

DLS 2: 0123456774561230371256046237045153016742104523764560712326743015 (DLS *B*, CF 2) – horizontal symmetry;

DLS 3: 0123456712407356735162405604371224761035351706246032547147652103 (DLS *C*, CF 3) – horizontal symmetry;

DLS 4: 0123456765370421371526047462513046507213204613755301674212743056 (DLS *D*, CF 4) – horizontal symmetry;

DLS 5: 0123456765370421376251047451623042107653104523762306174556743012 (DLS *E*, CF 3) – horizontal symmetry;

DLS 6: 0123456764570231376251047231645053107642104523764506172326743015 (DLS *F*, CF 5) – horizontal symmetry;

DLS 7: 0123456757143602153704267356124060452371246071354201675336725014 (DLS *G*, CF 6) – horizontal symmetry;

DLS 8: 0123456775361420371256046457023142016753104523762360714556743012 (DLS *H*, CF 5) – horizontal symmetry;

DLS 9: 0123456712043756735162405637041227461305451076236472503130652174 (DLS *I*, CF 7) – horizontal symmetry;

DLS 10: 0123456756043712735162406537042117452306426071532476103530125674 (DLS *J*, CF 8) – horizontal symmetry;

DLS 11: 0123456756407312735162406504372114752036326701542036147547125603 (DLS *K*, CF 7) – horizontal symmetry;

DLS 12: 0123456712043756735162405647031220361475451076236472503137652104 (DLS *L*, CF 5) – horizontal symmetry;

DLS 13: 0123456726043715735612406547032110352476426071535471603237125604 (DLS *M*, CF 2) – horizontal symmetry;

DLS 14: 0123456712407356753164204605271353761042241706356054327137625104 (DLS *N*, CF 4) – horizontal symmetry;

DLS 15: 0123456726407315735612406504372114752036326701545031647247125603 (DLS *O*, CF 5) – horizontal symmetry;

DLS 16: 0123456773652140405162732674301557361402640257313510762412470356 (DLS *P*, CF 6) – horizontal symmetry;

DLS 17: 0123456762370451371256047465123046507123104623755301674225743016
(DLS *Q*, CF 9);

DLS 18: 0123456762370451371256047456123045607123104523765301674226743015
(DLS *R*, CF 10) – horizontal symmetry;

DLS 19: 0123456762370451371256047465213046507213204613755301674215743026
(DLS *S*, CF 11) – horizontal symmetry;

DLS 20: 0123456762370451371256047456213045607213204513765301674216743025
(DLS *T*, CF 9);

DLS 21: 0123456712370456376251047415263046507213204613755301674265743021
(DLS *U*, CF 12) – horizontal symmetry;

DLS 22: 0123456715370426376521047412563046507213204613755301674262743051
(DLS *V*, CF 13) – horizontal symmetry;

DLS 23: 0123456715407326627430517412563053016742273614053657021440652173
(DLS *W*, CF 1) – horizontal symmetry;

DLS 24: 0123456775361420301256741457023642016753674523012360714556743012
(DLS *X*, CF 7) – horizontal symmetry;

DLS 25: 0123456763570241476521037541632024107635103254763206175456743012
(DLS *Y*, CF 7) – horizontal symmetry;

DLS 26: 0123456773561240401526731547032624016735673254013260715456743012
(DLS *Z*, CF 8) – horizontal symmetry;

DLS 27: 0123456726407315123704564765210353761042640257317514362030516274
(DLS *AA*, CF 14);

DLS 28: 0123456712307456754162302605371453761042341706256054237147625103
(DLS *AB*, CF 9);

DLS 29: 0123456712307456745162302604371553761042351706246045237147625103
(DLS *AC*, CF 10) – horizontal symmetry;

DLS 30: 0123456712307456754163203605271453761042241706356054327147625103
(DLS *AD*, CF 11) – horizontal symmetry;

DLS 31: 0123456712307456745163203604271553761042251706346045327147625103
(DLS *AE*, CF 9);

DLS 32: 0123456715407326723164504602571353761042241706356054327137652104
(DLS *AF*, CF 13) – horizontal symmetry;

DLS 33: 0123456715307426724163503602571453761042241706356054327147652103
(DLS *AG*, CF 12) – horizontal symmetry;

DLS 34: 0123456775461320367520144712560360543271246071355301674212370456
(DLS *AH*, CF 14) – horizontal symmetry.

Adjacency matrix:

0100000000000000000000000000000000
1010000000000000000000000000000000
0101111100000000000000000000000000
0010000011100000000000000000000000
0010000000011111000000000000000000
0010000000000000000000000000000000
0010000000000000000000000000000000
0010000000000000000000000000000000
0001000000000000000000000000000000
0001000000000000111111000000000000
0001000000000000000000000000000000
0000100000000000000000000000000000
0000100000000000000000100000000000
0000100000000000000000011100000000

00001000000000000000000000000000
00001000000000000000000000000000
00000000010000000000000000000000
00000000010000000000000010000000
00000000010000000000000000000000
00000000010000000000000000000000
00000000010000000000000000000000
00000000010000000000000000000000
00000000000100000000000000000000
00000000000010000000000000000000
00000000000010000000000000000000
00000000000010000000000000111110
00000000000000001000000000000000
00000000000000000000000100000000
00000000000000000000000100000001
00000000000000000000000100000000
00000000000000000000000100000000
00000000000000000000000100000000
00000000000000000000000100000000
00000000000000000000000000100000

Different CFs set within combinatorial structure:

CF 1: 0123456712307456406521732714360576425310350617245471603263570241 (DLS *A*, DLS *W*);

CF 2: 0123456723016745457610236754320110425376321706547635241054607132 (DLS *B*, DLS *M*);

CF 3: 0123456712370456674523017356124035107624406251735471603226043715 (DLS *C*, DLS *E*);

CF 4: 0123456712367405476503217054613235021674231057466471325056472013 (DLS *D*, DLS *N*);

CF 5: 0123456712043756357610242031647567425301741526305360714246570213 (DLS *F*, DLS *H*, DLS *L*, DLS *O*);

CF 6: 0123456723061745365702145764310212307456401256737541632064752031 (DLS *G*, DLS *P*);

CF 7: 0123456712043756357610245041637267325401741526302360714546570213 (DLS *I*, DLS *K*, DLS *X*, DLS *Y*);

CF 8: 0123456723016745451706236742530154607132763524103276105410543276 (DLS *J*, DLS *Z*);

CF 9: 0123456712307456607253417514062343516270376521045406173226473015 (DLS *Q*, DLS *T*, DLS *AA*, DLS *AB*, DLS *AE*);

CF 10: 0123456712307456604253714517062373516240376521045406173226743015 (DLS *R*, DLS *AC*);

CF 11: 0123456712307456637250417514362040516273376521045406173226470315 (DLS *S*, DLS *AD*);

CF 12: 0123456712043756356701246475203123107645503164727642531047561203 (DLS *U*, DLS *AG*);

CF 13: 0123456712307456504163723657021473625140471526032476103565043721 (DLS *V*, DLS *AF*);

CF 14: 0123456723017654526734011754602365721340741502364036517236402715 (DLS *AA*, DLS *AH*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 34,\ a = 33,\ \rho = \left[\underbrace{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}_{24}, 2, 2, 2, 2, 4, 4, 6, 6, 7, 7\right].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.17. Deer

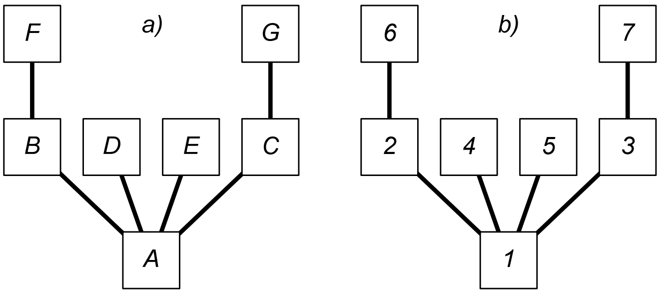

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712307654456723015674103274563210674501233012547623016745 (DLS *A*, CF 1) – vertical symmetry, column-inverse square;

DLS 2: 0123456764570213203176454765230156741032130267547546312032105476 (DLS *B*, CF 2) – vertical symmetry;

DLS 3: 0123456765740213203164757456230147651032130257465647312032107654 (DLS *C*, CF 3) – vertical symmetry;

DLS 4: 0123456764571023231076454765321056742301320167547546013210325476 (DLS *D*, CF 4) – vertical symmetry;

DLS 5: 0123456765741023231064757456321047652301320157465647013210327654 (DLS *E*, CF 5) – vertical symmetry;

DLS 6: 0123456746753120754603122301567432107456576412036457203110326745 (DLS *F*, CF 6) – vertical symmetry, column-inverse square;

DLS 7: 0123456774652031475631021032675432107645564713206574021323015476 (DLS *G*, CF 7) – vertical symmetry, column-inverse square.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712307654456723015674103274563210674501233012547623016745 (DLS *A*);

CF 2: 0123456712305746465703125476102367453201756421303012647523017654 (DLS *B*);

CF 3: 0123456712305746465721305476320167451023756403123012647523017654 (DLS *C*);

CF 4: 0123456712305746456723106475120357463021765401323012647523017654 (DLS *D*);

CF 5: 0123456712305746456701326475302157461203765423103012647523017654 (DLS *E*);

CF 6: 0123456712306745546723104675102375463201675401323012547623017654 (DLS *F*);

CF 7: 0123456712306745546721304675302175461203675403123012547623017654
(DLS *G*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$v = 7,\ a = 6,\ \rho = [1, 1, 1, 1, 2, 2, 4]$.

Method of finding:
Brute Force + Euler-Parker method + DLX.

7.18. Pyramid-1-4-3

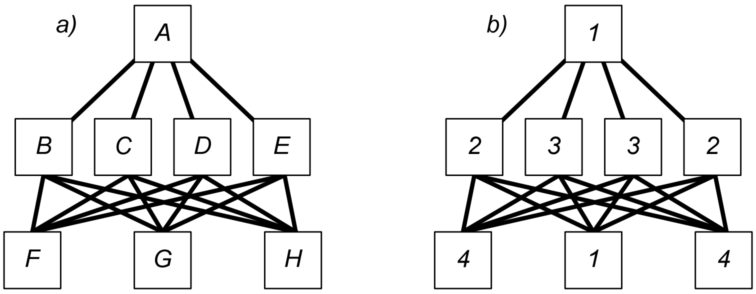

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456712307456547610324765210323016745604253717654321035170624 (DLS *A*, CF 1) – horizontal symmetry;

<u>DLS 2:</u> 0123456724061735765432105371604217352406351706246042537142607153 (DLS *B*, CF 2) – horizontal symmetry, string inverse;

<u>DLS 3:</u> 0123456723016745765432105476103247652103153704263012567462407351 (DLS *C*, CF 3) – horizontal symmetry, string inverse;

<u>DLS 4:</u> 0123456754761032765432102301674512307456371526046547032140625173 (DLS *D*, CF 3) – horizontal symmetry, string inverse;

<u>DLS 5:</u> 0123456753716042765432102406173542607153173524063517062460425371 (DLS *E*, CF 2) – horizontal symmetry, string inverse;

<u>DLS 6:</u> 0123456762407351547610323715260426043715406251737351624015370426 (DLS *F*, CF 4) – horizontal symmetry;

<u>DLS 7:</u> 0123456765470321547610323012567476543210426071532301674517352406 (DLS *G*, CF 1) – horizontal symmetry;

<u>DLS 8:</u> 0123456715370426547610324062517373516240624073512604371537152604 (DLS *G*, CF 4) – horizontal symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456712307456547610324765210323016745604253717654321035170624 (DLS *A*, DLS *G*);

<u>CF 2:</u> 0123456712307654654721035476301276541230476503213012547623016745 (DLS *B*, DLS *E*);

<u>CF 3:</u> 0123456712307456765432106547032123016745406251735476103237152604 (DLS *C*, DLS *D*);

<u>CF 4:</u> 0123456712307456456701237654321030125674230167455476103267452301 (DLS *F*, DLS *H*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 8,\ a = 16,\ \rho = [4, 4, 4, 4, 4, 4, 4, 4].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.19. Tree-2

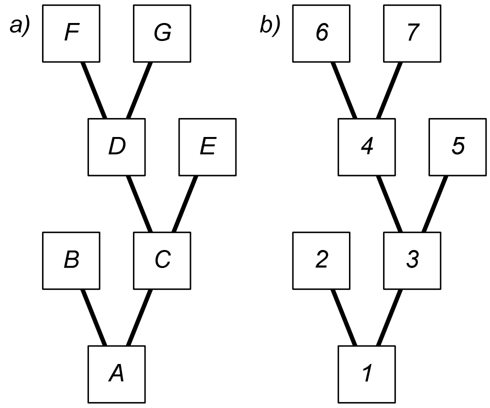

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456712307456356701245604371274561230674523014012567323716045
(DLS *A*, CF 1) – horizontal symmetry;

<u>DLS 2:</u> 0123456726743015603254713715260412407356456701237351624054061732
(DLS *B*, CF 2) – horizontal symmetry;

<u>DLS 3:</u> 0123456756470312203614757451623047652103130257466274305135107624
(DLS *C*, CF 3) – horizontal symmetry;

<u>DLS 4:</u> 0123456757361402456701232041637563025741765432101475203632107654
(DLS *D*, CF 4) – horizontal symmetry, string inverse;

<u>DLS 5:</u> 0123456774516230357610244602571350643172231076451247035667352401
(DLS *E*, CF 5) – horizontal symmetry;

<u>DLS 6:</u> 0123456713507246674253017236145036752014456701235014367224016735
(DLS *F*, CF 6) – horizontal symmetry;

<u>DLS 7:</u> 0123456736570214624073511475203657361402230167457514362040625173
(DLS *G*, CF 7) – horizontal symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456712307456356701245604371274561230674523014012567323716045
(DLS *A*);

<u>CF 2:</u> 0123456712307456456701235674301273561240674253013015267424016735
(DLS *B*);

<u>CF 3:</u> 0123456712043756365702144576102323607145573164027412563060452371
(DLS *C*);

<u>CF 4:</u> 0123456712370456371256047654321054016732654073214065217323761045
(DLS *D*);

CF 5: 0123456712643750305762144576102354107632264103757305214667325401 (DLS *E*);

CF 6: 0123456712043756356701244751620356307412604253717315264024761035 (DLS *F*);

CF 7: 0123456723016745571436027536142064725031465072131045237632670154 (DLS *G*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 7, a = 6, \rho = [1, 1, 1, 1, 2, 3, 3]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

Remark:

for different orders there are other types of trees, invariant from number of vertices, arcs and sorted powers of vertices is matches with invariant for Tree-1 structure from DLSs of order 10, but corresponding graphs are don’t isomorphic.

7.19. Tree-3

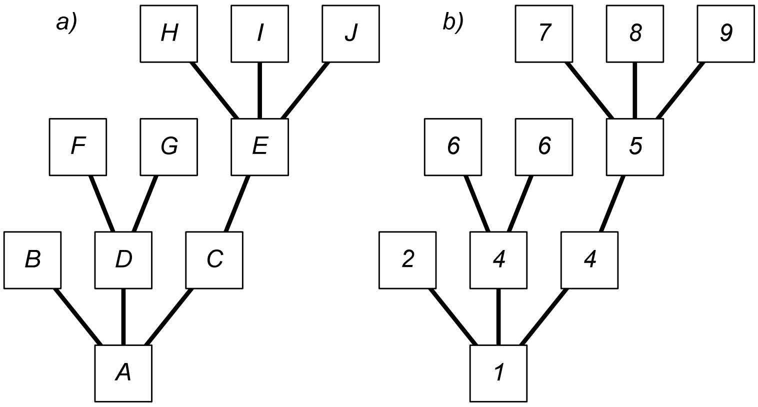

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456712305746456721036745301254761230765403213012647523017654 (DLS *A*, CF 1) – vertical symmetry;

DLS 2: 0123456754672310205476317631024567023154134567024576102332105476 (DLS *B*, CF 2) – horizontal symmetry;

DLS 3: 0123456754672310673102542076543113547602450231767645102332106745 (DLS *C*, CF 3) – horizontal symmetry;

DLS 4: 0123456745762310763102542054673113457602670231455467102332105476 (DLS *D*, CF 4) – horizontal symmetry;

DLS 5: 0123456745761023234576017601325467102345325467105467013210325476 (DLS *E*, CF 5) – vertical symmetry, column-inverse square;

DLS 6: 0123456713475206356471206415307250762413265017347201634547320651 (DLS *F*, CF 6) – vertical symmetry;

DLS 7: 0123456714357206206451734207361567521430754603213610275453716042 (DLS *G*, CF 6) – vertical symmetry;

DLS 8: 0123456754670132673210543276540110547623451032767645231023016745 (DLS *H*, CF 7) – vertical symmetry;

DLS 9: 0123456776450132643210753257640110645723571032464576231023017654 (DLS *I*, CF 8) – vertical symmetry;

DLS 10: 0123456776452310643102752057643113645702570231464576102332107654 (DLS *J*, CF 9) – vertical symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456712305746456721036745301254761230765403213012647523017654 (DLS *A*);

CF 2: 0123456712307456475162032647031553061742751436206472503130652174 (DLS *B*);

CF 3: 0123456723061745326701547514362017325406465072135041637264752031 (DLS *C*);

CF 4: 0123456723016745154073265736140274625130365702144015267362743051 (DLS *D*);

CF 5: 0123456712307456765432106547032123761045401526735401673237625104 (DLS *E*);

CF 6: 0123456712043756273614055467013260425371731526404650721335716024 (DLS *F*, DLS *G*);

CF 7: 0123456712057634736410255071234624365701371064526542317046570213 (DLS *H*);

CF 8: 0123456712307456471256033567012474516230604523715306174226743015 (DLS *I*);

CF 9: 0123456712043756357610245731640260425371741526302360714546570213 (DLS *J*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 10,\ a = 9,\ \rho = \left[1, 1, 1, 1, 1, 1, 2, 3, 3, 4\right].$

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.20. Cross

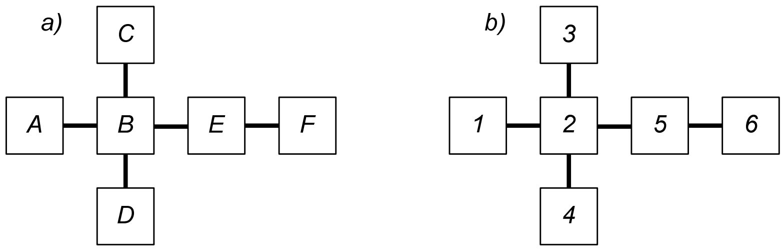

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1:</u> 0123456723061745154073266735240174516230326701544012567356743012 (DLS *A*, CF 1);
<u>DLS 2:</u> 0123456754716032371526042306174515307426765432106247035140625173 (DLS *B*, CF 2);
<u>DLS 3:</u> 0123456763025741154073265731640274652130321706544056127326743015 (DLS *C*, CF 3);
<u>DLS 4:</u> 0123456767425301153704265471603240652173321076547356124026043715 (DLS *E*, CF 4);
<u>DLS 5:</u> 0123456727461305153704266475203140516273326071547312564056043712 (DLS *D*, CF 5);
<u>DLS 6:</u> 0123456746052713735162406742530115307426206431755417063232761054 (DLS *F*, CF 6).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 0123456723061745154073266735240174516230326701544012567356743012 (DLS *A*);
<u>CF 2:</u> 0123456712370456765432102476103565407321376251045301674240152673 (DLS *B*);
<u>CF 3:</u> 0123456723061745475162036574302112407356301256747635241054670132 (DLS *C*);
<u>CF 4:</u> 0123456712357604706413255701243623465071341067526572314046570213 (DLS *E*);
<u>CF 5:</u> 0123456712346705341652706705341223417056705216345670234145670123 (DLS *D*);
<u>CF 6:</u> 0123456723061745471526037564312036725014645072315031647212470356 (DLS *E*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 6, a = 5, \rho = [1, 1, 1, 1, 2, 4]$.

Method of finding:

Brute Force + Euler-Parker method + DLX.

7.21. Huge structure based on DLS with 824 ODLS (N824HUGE)

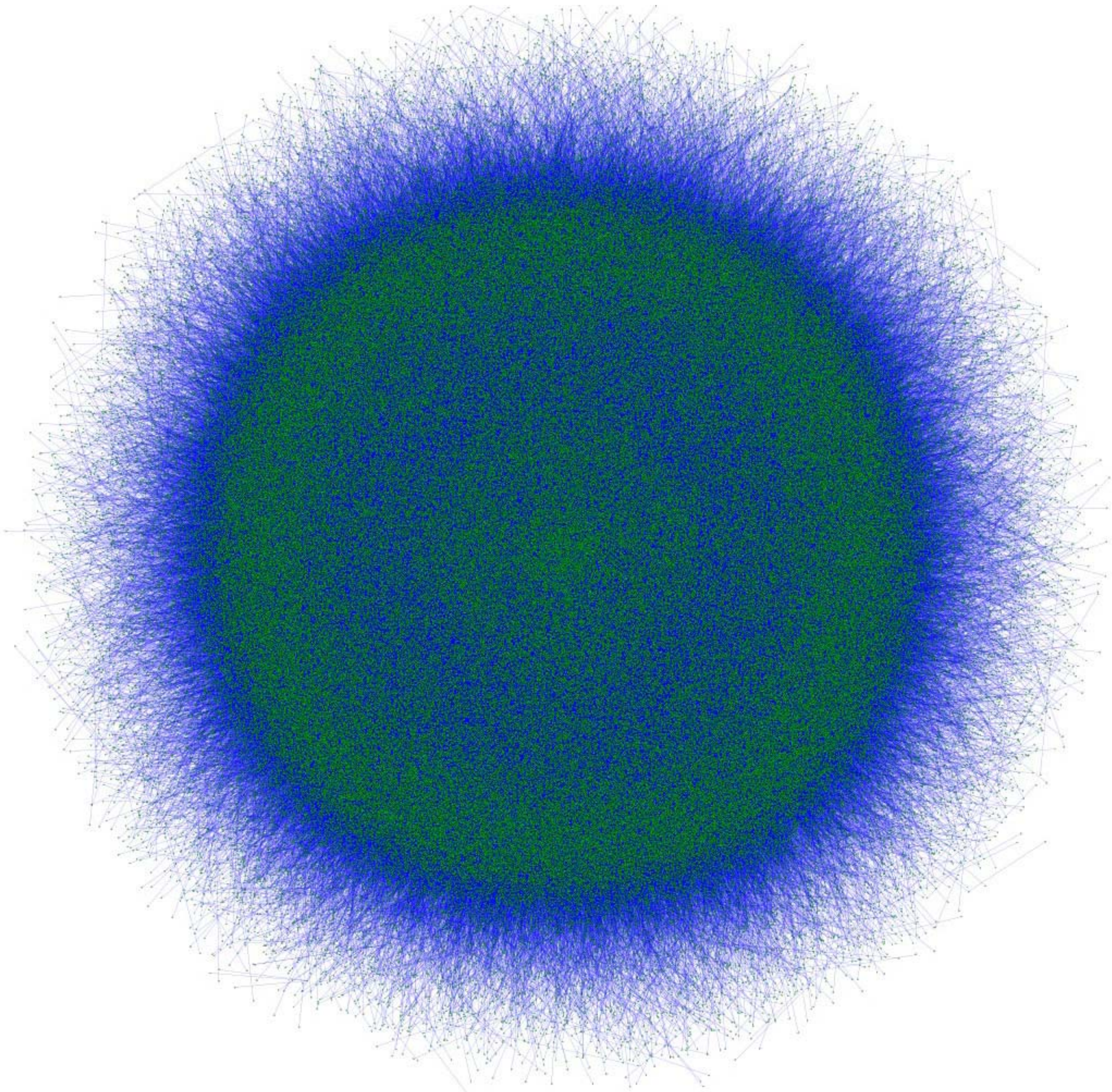

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure (348 000 DLSs):

DLS 1: 0123456723061745154073266735240174516230326701544012567356743012;
DLS 2: 0123456754716032371526042306174515307426765432106247035140625173;
…
DLS 348 000: 0123456746052713735162406742530115307426206431755417063232761054.

Different CFs set within combinatorial structure (657 items):

CF 1: 0123456723016745456701236745230132107654103254767654321054761032;
CF 2: 0123456723016745456710326745321054762301765401233210547610327654;
…
CF 657: 0123456712057436437260513016572454613270654721037650134227340615.

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 348\,000$, $a = 482\,688$,

$$\rho = [\underbrace{1, 1, ..., 1}_{270\,528}, \underbrace{2, 2, ..., 2}_{24\,480}, \underbrace{3, 3, ..., 3}_{5\,952}, \underbrace{4, 4, ..., 4}_{28\,800}, \underbrace{5, 5, ..., 5}_{1152}, \underbrace{6, 6, ..., 6}_{1\,344}, \underbrace{7, 7, ..., 7}_{3\,456}, \underbrace{8, 8, ..., 8}_{576},$$

$$\underbrace{9, 9, ..., 9}_{384}, \underbrace{10, 10, ..., 10}_{288}, \underbrace{11, 11, ..., 11}_{384}, \underbrace{12, 12, ..., 12}_{768}, \underbrace{13, 13, ..., 13}_{384}, \underbrace{14, 14, ..., 14}_{864},$$

$$\underbrace{16, 16, ..., 16}_{1\,056}, \underbrace{18, 18, ..., 18}_{576}, \underbrace{19, 19, ..., 19}_{576}, \underbrace{20, 20, ..., 20}_{1\,440}, \underbrace{22, 22, ..., 22}_{576}, \underbrace{24, 24, ..., 24}_{384},$$

$$\underbrace{28, 28, ..., 28}_{1\,344}, \underbrace{40, 40, ..., 40}_{96}, \underbrace{45, 45, ..., 45}_{576}, \underbrace{48, 48, ..., 48}_{96}, \underbrace{50, 50, ..., 50}_{288}, \underbrace{108, 108, ..., 108}_{768},$$

$$\underbrace{116, 116, ..., 116}_{288}, \underbrace{128, 128, ..., 128}_{96}, \underbrace{131, 131, ..., 131}_{384}, \underbrace{824, 824, ..., 824}_{96}].$$

Method of finding:

Brute Force + Euler-Parker method + DLX.

Remark:

Structure contains clique from more than 2 DLSs unlike all other structures from DLSs of order 8. Clique consists from 6 MODLS [1]:

DLS 1: 0123456723016745456701236745230132107654103254767654321054761032 (CF 1);
DLS 2: 0123456732107654674523015476103276543210456701231032547623016745 (CF 1);
DLS 3: 0123456745670123321076547654321067452301230167455476103210325476 (CF 1);
DLS 4: 0123456754761032103254764567012323016745765432103210765467452301 (CF 1);
DLS 5: 0123456767452301765432101032547654761032321076542301674545670123 (CF 1);
DLS 6: 0123456776543210547610322301674510325476674523014567012332107654 (CF 1).

All MODLS within clique have same CF:

CF 1: 0123456723016745456701236745230132107654103254767654321054761032.

7.22. Different ODLS of order 8 CFs set with corresponding combinatorial structures

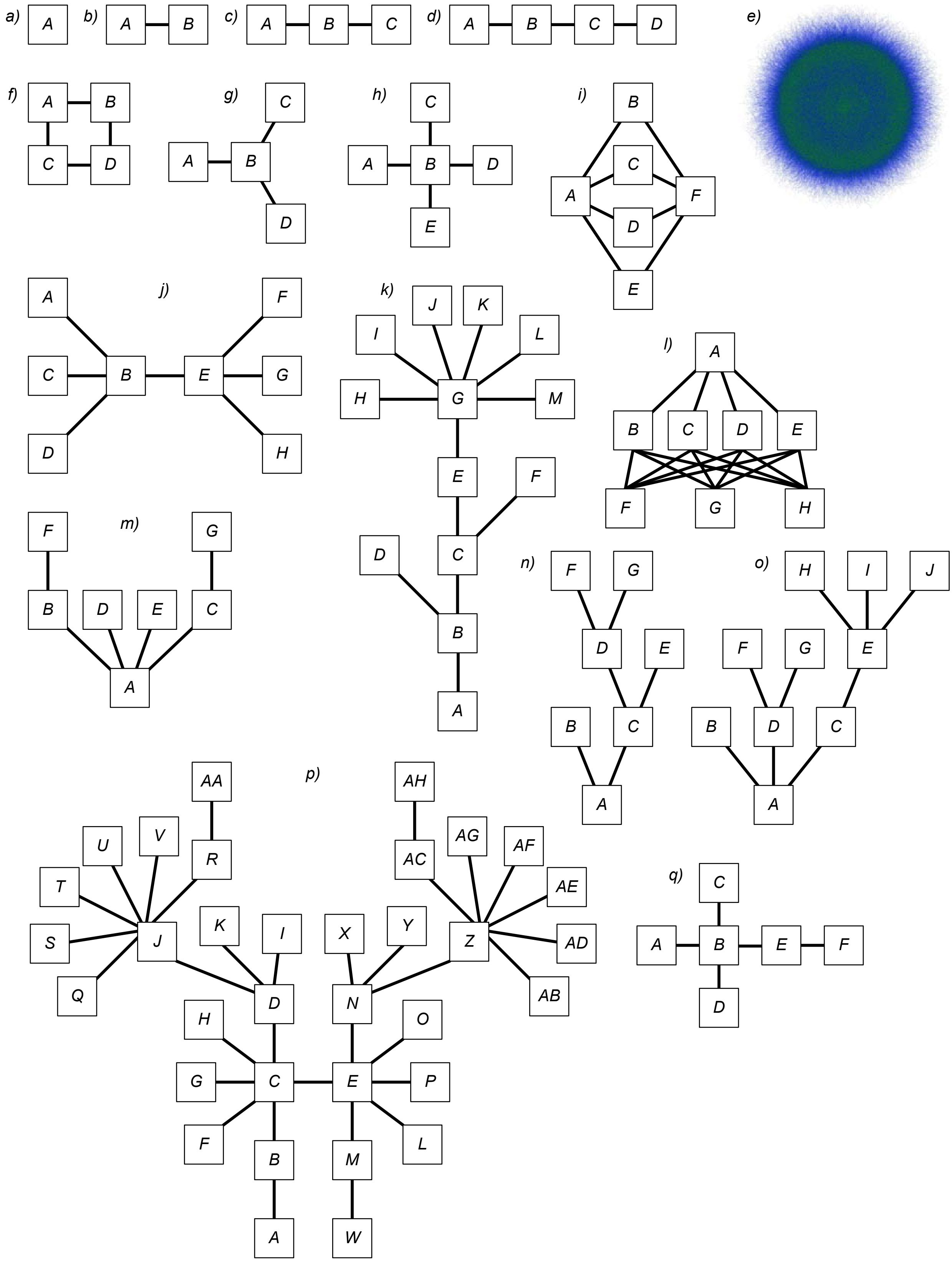

Рис. Different ODLS of order 8 CFs set with corresponding combinatorial structures

ONCE (A):1 – 267, where:
1 CFs – 1

2 CFs – 266

LINE3 (B):1 – 28, where:
2 CFs – 4
3 CFs – 24

LINE3 (B):2 – 16, 17:1, where:
2 CFs – 4
3 CFs – 12

LINE4 (C):1 – 4, where:
4 CFs – 4

LINE4 (C):2 – 4, where:
4 CFs – 4

LOOP4 (E):2 – 1, where:
1 CFs – 1

1TO3 (F):1 – 4, where:
3 CFs – 4

1TO3 (F):3 – 2, where:
3 CFs – 2

1TO4 (G):1 – 25, where:
3 CFs – 10
4 CFs – 3
5 CFs – 12

1TO4 (G):4 – 9, where:
3 CFs – 5
4 CFs – 1
5 CFs – 3

RHOMBUS4 (K):2 – 2, where:
3 CFs – 2

RHOMBUS4 (K):4 – 1, where:
3 CFs – 1

HEDGEHOG3 (Q):1 – 10, where:
6 CFs – 4
8 CFs – 6

HEDGEHOG3 (Q):4 – 4, where:
6 CFs – 2
8 CFs – 2

DANDELION (R):1 – 8, where:
12 CFs – 8

DANDELION (R):2 – 1, where:
12 CFs – 1

DANDELION (R):3 – 2, where:
12 CFs – 2

DANDELION (R):7 – 1, where:
12 CFs – 1

CACTUS (S):1 – 9, where:
14 CFs – 9

CACTUS (S):2 – 2, where:
14 CFs – 2

CACTUS (S):4 – 1, where:
14 CFs – 1

CACTUS (S):6 – 1, where:
14 CFs – 1

CACTUS (S):7 – 1, where:
14 CFs – 1

DEER (T):1 – 4, where:
7 CFs – 4

DEER (T):2 – 2, where:
7 CFs – 2

DEER (T):4 – 1, where:
7 CFs – 1

N8PYRAMID143 (U):4 – 16, where:
2 CFs – 4
4 CFs – 12

TREE2 (V):1 – 4, where:
7 CFs – 4

TREE2 (V):2 – 1, where:
7 CFs – 1

TREE2 (V):3 – 2, where:
7 CFs – 2

TREE3 (W):1 – 5, where:
9 CFs – 5

TREE3 (W):2 – 1, where:
9 CFs – 1

TREE3 (W):3 – 2, where:
9 CFs – 2

TREE3 (W):4 – 1, where:
9 CFs – 1

CROSS (X):1 – 4, where:
6 CFs – 4

CROSS (X):2 – 1, where:
6 CFs – 1

CROSS (X):4 – 1, where:
6 CFs – 1

N824HUGE:1 – 414
N824HUGE:2 – 63
N824HUGE:3 – 16
N824HUGE:4 – 77
N824HUGE:5 – 6
N824HUGE:6 – 4
N824HUGE:7 – 9
N824HUGE:8 – 2
N824HUGE:9 – 2
N824HUGE:10 – 2
N824HUGE:11 – 2
N824HUGE:12 – 4
N824HUGE:13 – 1
N824HUGE:14 – 5
N824HUGE:16 – 9
N824HUGE:18 – 3
N824HUGE:19 – 3
N824HUGE:20 – 5
N824HUGE:22 – 3
N824HUGE:24 – 2
N824HUGE:28 – 10
N824HUGE:40 – 1
N824HUGE:45 – 3
N824HUGE:48 – 1
N824HUGE:50 – 2
N824HUGE:108 – 3
N824HUGE:116 – 2
N824HUGE:128 – 1
N824HUGE:131 – 1
N824HUGE:824 – 1

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